دانشگاه آزاد
مرودشت

Mathematical Foundations

Linear algebra is the natural mathematical language of **quantum mechanics**

Hilbert spaces for vectors and operators

The Dirac notation

A Hermitian (self-adjoint) operator produces a basis set within a Hilbert space

Observables such as energy or momentum

By: Dr. Jabbari

In Q.M.:

Act of observing represented by operators

Particle or system represented by vectors

Exam: ψ (wave funtion) or |ψ> (Ket) represents particle moving along $\hat{z}$ axis unlike Movement p . Let p be an operator that measures momentum

$$\hat{p}\,|\psi\rangle = p\,|\psi\rangle \qquad p = \text{result of measurement}$$

Exam: Alternate statement (Average momentum)

$$\langle \hat{p} \rangle = \text{average momentum} = \langle \psi | \hat{p} | \psi \rangle = p \langle \psi | \psi \rangle = p$$

کمیت (Quantities) مشاهده پذیر (Observable) توسط یک عملگر هرمیتی نمایش داده می شود

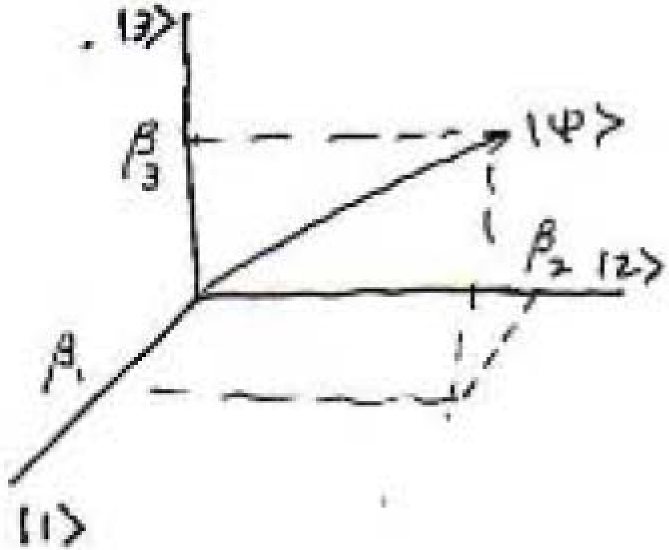
$$\text{Energy} = \hat{H} \qquad \text{momentum} = \hat{p} \qquad \text{Angul. mom } \hat{L}$$

Basic problem in QM. is

Wave functions live in Hilbert space

Hilbert space = vector space + Inner product

There exists basis vectors



$$|\psi\rangle = \sum_i \beta_i \, |i\rangle$$

Function   function

$$\psi = \sum_i \beta_i \, \psi_i(x)$$

Usually basis set $\{ \psi_i \sim |\psi_i\rangle \sim |i\rangle \}$ chosen to be eigen vectors of observable like $\hat{H}$ ~ Hamilton

$$\hat{O} \, |\psi_i\rangle = o_i \, |\psi_i\rangle$$

By: Dr. Jabbari

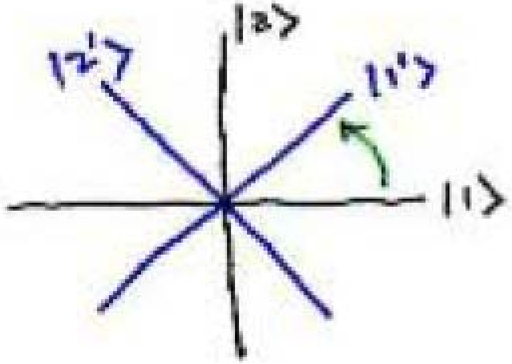
- β_i = component = probability Amplitude

$= \langle \psi_i | \psi \rangle = \langle i | \psi \rangle$ = Probability Amplitude

Related to Probability of finding particle with wave function ψ in Basic State ψ_i

$$P(i) = |\beta_i|^2 = \langle \psi_i | \psi \rangle = \int dx \, \psi_i^* \psi$$

Other type of operator : unitary converts one basis Set into another



+ "Derive" (i.e. Rediscover) Schrodingers Equation

$\hat{\mathcal{H}}\psi = i\hbar\frac{\partial\psi}{\partial t}$: time dependent

$\hat{\mathcal{H}}\psi_E = E\psi_E$: time independent gives basis vectors

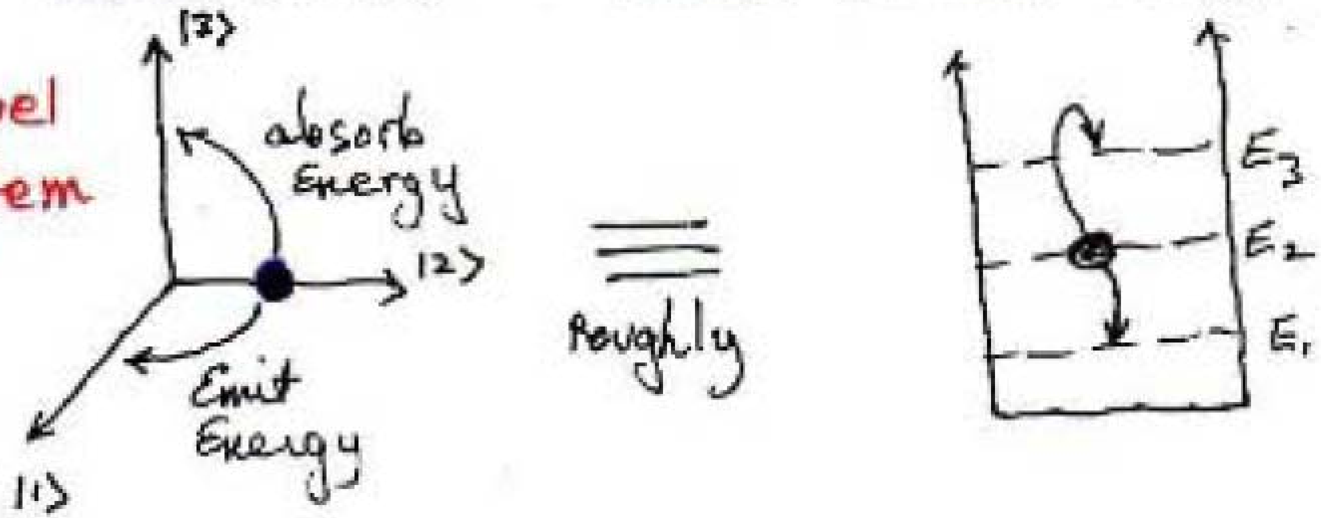
• will need to Review partial Differential Equations

$\hat{\mathcal{H}}\psi = i\hbar\frac{\partial\psi}{\partial t}$

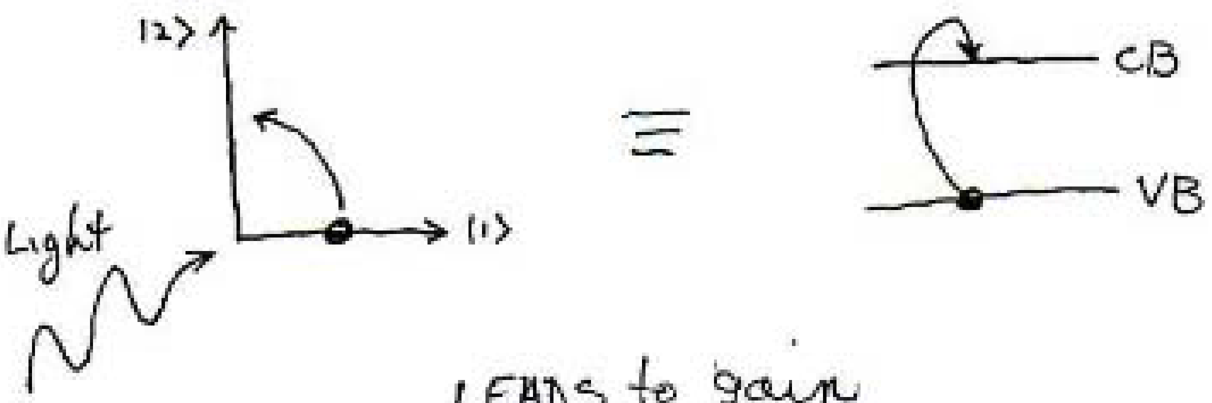
$\hat{\mathcal{H}}\psi_E = E\psi_E$

+ Discuss perturbation theory and transitions : Fermi Golden Rule

3 Level System



+ Two levels



LEADS to gain

- + Operators Also form a vector space (and Hilbert Space)
- Operators or Wae functions can carry the dynamics of system
- ·Representations Schrodinger ,Heisenberg ,Interaction

Definition of Vector Space

vector space consists

- set **F**(**real or complex**)with operator $+$ f, f_1, f_2 in F
- Scalar multiplication (SM) over the field of numbers $\mathcal{N}$
  α, β, γ in $\mathcal{N}$

Closure:	f_1+f_2 is in F and αf is in F
Associative:	$(f_1+f_2)+f_3=f_1+(f_2+f_3)$
Commutative:	$f_1+f_2=f_2+f_1$
Zero:	There exists a zero vector $\mathcal{O}$ such that $\mathcal{O}+f=f$
Negatives:	For every f in F, there exists $(-f)$ in F such that $f+(-f)=\mathcal{O}$
SM Associative:	$(\alpha\beta)f=\alpha(\beta f)$
SM Distributive:	$\alpha(f_1+f_2)=\alpha f_1+\alpha f_2$
SM Distributive:	$(\alpha+\beta)f=\alpha f+\beta f$
SM Unit:	$1f=f$

Example:

If **F** represents the set of **real** functions but the **number** field consists of **complex numbers**

Objects such as $c_1 f(x)$ (where c_1 is complex) cannot be in the original vector space because the function g(x)=c1f(x) has complex values.

closure cannot be satisfied

cannot be vector space

Inner Product, Norm, and Metric

An inner product $\langle \bullet | \bullet \rangle$ in a (real or complex) vector space F is a scalar valued function that maps $F \times F \to C$ (where C is the set of complex numbers) with the properties

1. $\langle f | g \rangle = \langle g | f \rangle^*$ with f, g elements in F and where "*" denotes complex conjugate.
2. $\langle \alpha f + \beta g | h \rangle = \alpha^* \langle f | h \rangle + \beta^* \langle g | h \rangle$ and $\langle h | \alpha f + \beta g \rangle = \alpha \langle h | f \rangle + \beta \langle h | g \rangle$ where f, g, h are elements of F and α, β are elements in the complex number field $\mathscr{C}$.
3. $\langle f | f \rangle \geq 0$ for all vectors f. The inner product can be zero $\langle f | f \rangle = 0$ if and only if $f = 0$ (except at possibly a few points for functions).

The **norm** or **"length"** of a vector f is defined to be $\|f\| = \langle f \mid f \rangle^{1/2}$

A **metric d(f, g)** is a relation between two elements f and g of a set F such that

1. $d(f, g) \geq 0$ and $d = 0$ only when $f = g$ (except at possibly a few points for piecewise continuous functions $C_p[a,b]$). Recall that two functions are equal only when $f(x) = g(x)$ for all "x" in the domain of definition.
2. $d(f, g) = d(g, f)$.
3. $d(f, g) \leq d(f, h) + d(h, g)$ where h is any third element of F.

The **metric measures** the distance between two elements of the space

$$d(f,g) = \langle f - g \mid f - g \rangle^{1/2}$$

$$\vec{r}_1 = x_1\tilde{x} + y_1\tilde{y}$$

در فضای اقلیدسی R^2

$$\vec{r}_2 = x_2\tilde{x} + y_2\tilde{y}$$

Inner product $\langle \vec{r}_1 | . \vec{r}_2 \rangle = \vec{r}_1 \cdot \vec{r}_2 = x_1x_2 + y_1y_2$

Norm $\|\vec{r}_1\| = \langle \vec{r}_1 | \vec{r}_1 \rangle^{1/2} = (x_1^2 + y_1^2)^{1/2}$

Metric $d(\vec{r}_1, \vec{r}_2) = \|\vec{r}_1 - \vec{r}_2\| = [(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2)]^{1/2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

inner product for functions

Inner product

Norm $\|f(x)\| = \langle f | f \rangle^{1/2} = \left[\int_a^b dx\, f(x)^* f(x)\right]^{1/2} = \left[\int_a^b dx\, |f(x)|^2\right]^{1/2}$

Example: Find the length of f(x)=x for x ε[1, 1]

$$\|f\| = \langle f \mid f \rangle^{1/2} = \left[\int_{-1}^{1} dx\, x^* x\right]^{1/2} = \left[\int_{-1}^{1} dx\, x \cdot x\right]^{1/2} = \left[\int_{-1}^{1} dx\, x^2\right]^{1/2} = \sqrt{\frac{2}{3}}$$

Kets, Bras, and Brackets for Euclidean Space

3D euclidean vector: $\{\tilde{x}, \tilde{y}, \tilde{z}\}$

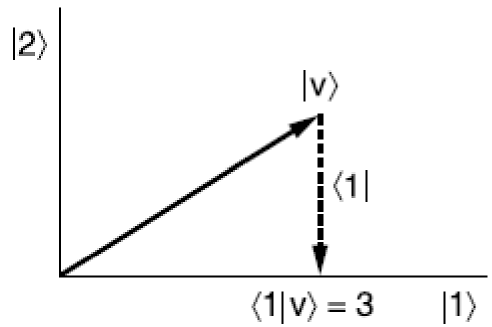
$$\tilde{x} \leftrightarrow |1\rangle \quad \tilde{y} \leftrightarrow |2\rangle \quad \tilde{z} \leftrightarrow |3\rangle \qquad \tilde{e}_n \leftrightarrow |n\rangle$$

Example: $\vec{v} = 3\tilde{x} - 4\tilde{y} + 10\tilde{z}$

$$|v\rangle = 3|1\rangle - 4|2\rangle + 10|3\rangle$$

We define a **"bra" < |** to be a projection operator

$$|v\rangle = 3|1\rangle - 4|2\rangle + 10|3\rangle \quad \Rightarrow \quad \begin{cases} \langle 1|\vec{v} = 3 \\ \langle 2|\vec{v} = -4 \\ \langle 3|\vec{v} = 10. \end{cases}$$



Projection of $\vec{v} = 3\tilde{x} + 5\tilde{y}$ onto $|1\rangle, |2\rangle$.

Combination of projection operators and vectors as

- Already Know
  $\langle w| \equiv \vec{w}\,\cdot$

$$\langle 1|\vec{v} = \langle 1|v\rangle$$

"bra" + "ket" gives the "braket" **inner product**

$$\langle w|v\rangle = (\vec{w}\cdot)\vec{v} = \vec{w}\cdot\vec{v}.$$

$|\bullet\rangle\langle\bullet|$ → **to be more complicated compound objects**

دانشگاه آزاد
مرودشت

Example $\vec{v} = 3\hat{x} + 4\hat{y} - 5\hat{z}$

$$\langle 1|v\rangle = \langle 1|\,|v\rangle = \hat{x}\cdot(3\hat{x}+4\hat{y}-5\hat{z}) = 3$$

or

$$\langle 1|v\rangle = \langle 1|\,|v\rangle = \langle 1|\{3|1\rangle + 4|2\rangle - 5|3\rangle\}$$

$$= 3\langle 1|1\rangle + 4\langle 1|2\rangle - 5\langle 1|3\rangle = 3$$

+ mention Dual Vector Space (more later)

- $\langle w|$ = Linear Operator maps $V \rightarrow \mathbb{C}$

$$\langle w|\sum_{i=1}^{n} c_i |v_i\rangle = \sum_{i=1}^{n} c_i \langle w|v_i\rangle$$

- For every Vector $|w\rangle$ there is an operator $\langle w|$, that is

$$\langle w| \longleftrightarrow |w\rangle \quad \text{or} \quad \vec{W}\cdot \longleftrightarrow \vec{W}$$

By: Dr. Jabbari

Basis, Completeness, and Closure for Euclidean Space

basis set must be orthonormal and complete

$$\langle m|n\rangle = \delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Kronecker delta function

$$\langle 1|2\rangle = 0 = \langle 1|3\rangle = \cdots = \langle m|n\rangle \quad \text{for } m \neq n$$

A linear combination of ''N'' orthonormal vectors $B = \{|1\rangle, |2\rangle, \ldots, |N\rangle\}$ has the form

$$|v\rangle = \sum_{i=1}^{N} C_i |i\rangle$$

can be complex numbers

closure (i.e., completeness) relation

$$\langle m \mid v\rangle = \langle m| \sum_{i=1}^{n} C_i |i\rangle = \sum_{i=1}^{n} C_i \langle m \mid i\rangle = \sum_{i=1}^{n} C_i \delta_{i,m} = C_m$$

$$C_i = \langle i \mid v\rangle$$

$$|v\rangle = \sum_{i=1}^{n} C_i |i\rangle = \sum_{i=1}^{n} [\langle i \mid v\rangle] |i\rangle$$

$$|v\rangle = \sum_{i=1}^{n} |i\rangle\langle i \mid v\rangle \quad \text{or} \quad |v\rangle = \left(\sum_{i=1}^{n} |i\rangle\langle i|\right)|v\rangle$$

$$\sum_{i=1}^{n} |i\rangle\langle i| = 1$$

Example 4.2.1

The completeness relation for $\mathscr{R}^3$ using $\langle w| = \vec{w}\cdot$ is

$$1 = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \quad \text{so} \quad 1 = \tilde{x}\,\tilde{x}\cdot + \tilde{y}\,\tilde{y}\cdot + \tilde{z}\,\tilde{z}\cdot$$

Note that the unit vectors are written next to each other without an operator between them.

The Euclidean Dual Vector Space

Bra projects an arbitrary vector onto the vector

linear operator $\langle w|$ maps a vector space V into the complex numbers $\mathscr{C}$ (i.e., $\langle w|: V \to \mathscr{C}$).

Euclidean vector ,the corresponding bra is the operator $|v\rangle^{+} = \langle v| = \vec{v}\cdot$.

$$\langle\cdot| \quad \overset{+}{\leftrightarrow} \quad |\cdot\rangle \quad \text{or as} \quad |w\rangle^{+} = \langle w|$$

"dual vector space V^+

$V = \{\, |v\rangle \,\}$ and $V^{+} = \{\, \langle w| \,\}$

Example 4.2.2

Find the vector dual to $|2\rangle = \hat{y}$.
The dual vector is $\langle 2| = \hat{y}\cdot$ which is an operator that projects an arbitrary vector $\vec{v}$ onto $\hat{y}$.
We can explicitly represent the result of the projection as the y-component of $\vec{v}$:

$$\langle 2 \mid v\rangle = \hat{y}\cdot\vec{v} = v_y$$

Example 4.2.3

Some relations can be demonstrated for $\vec{v} = |v\rangle = a|1\rangle + b|2\rangle$ where $\{\, |1\rangle, |2\rangle \,\}$ spans R^2.

1. $\langle v| = |v\rangle^{+} = [a|1\rangle + b|2\rangle]^{+} = \left[|1\rangle^{+}a^{+} + |2\rangle^{+}b^{+}\right] = a^*\langle 1| + b^*\langle 2|$

2. $\langle v \mid 1\rangle = [a^*\langle 1| + b^*\langle 2|]|1\rangle = a^*$ and $\langle 1 \mid v\rangle = \langle 1|[a|1\rangle + b|2\rangle] = a$

3. $\langle 1 \mid v\rangle = a = (a^*)^* = \langle v \mid 1\rangle^*$ Note that $\langle v \mid 1\rangle^{+} = \langle 1 \mid v\rangle = \langle v \mid 1\rangle^*$.

The adjoint reverses the order of operators

$$\langle v|L_1 L_2|w\rangle^+ = \langle w|L_2^+ L_1^+|v\rangle$$

Exam: Assume $\{|i\rangle : \ i = 1, 2, 3\}$

norm

$$\|\vec{v}\|^2 = \langle v \mid v\rangle = \left(\sum_{i=1}^{3} v_i|i\rangle\right)^+ \left(\sum_{j=1}^{3} v_j|j\rangle\right) = \sum_{i=1}^{3} \langle i|v_i^* \sum_{j=1}^{3} v_j|j\rangle = \sum_{i,j=1}^{3} \langle i|v_i^* v_j|j\rangle = \sum_{i,j=1}^{3} v_i^* v_j\langle i \mid j\rangle$$

unit vectors

$$\|v\|^2 = \sum_{i,j=1}^{3} v_i^* v_j \delta_{i,j} = \sum_{i=1}^{3} v_i^* v_i = \sum_{i=1}^{3} |v_i|^2$$

the magnitude of the complex number

Hilbert Space

A Hilbert space consists of a vector space of functions with a defined inner product

Hilbert Space of Functions with Discrete Basis Vectors

Functions in a set $F = \{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\}$ are **linearly independent** if for complex constants

$$\sum_{i=0}^{n} c_i \phi_i(x) = 0$$

can only be true when all of the complex constants are zero $c_i = 0.$

Functions in the set $F = \{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\}$ are **orthonormal** if $\langle \phi_i \mid \phi_j \rangle = \delta_{ij}$

A linearly independent set of functions F is **complete** if every function f(x) in the space can be written as

$$f(x) = \sum_{i=0}^{n} c_i \phi_i(x) \quad \text{or} \quad |f\rangle = \sum_{i=0}^{n} c_i |\phi_i\rangle$$

the set $\{\phi_i\}$ complete and **orthonormal** → **can be chosen as basis functions (or basis vectors) to span the function space and basis for a Hilbert space**

$$\{|\phi_0\rangle, |\phi_1\rangle, \ldots\} = \{|0\rangle, |1\rangle, \ldots\}$$

Then we can write each function to form $|f\rangle = \sum_{i=0}^{\infty} c_i |i\rangle$

where the components of the vector can be find as follow:

$$\langle j \mid f\rangle = \langle \phi_j \mid f\rangle = \langle j| \sum_{i=0}^{\infty} c_i |i\rangle = \sum_{i=0}^{\infty} c_i \langle j \mid i\rangle = \sum_{i=0}^{\infty} c_i \delta_{ij} = c_j$$

The **projection** of the function on **the i^{th} axis** produces the inner product between the two complex functions

$$\langle \phi_i \mid f\rangle = \int_a^b dx\, \phi_i^*(x) f(x)$$

$$|f\rangle = \sum_{i=0}^{\infty} c_i |i\rangle$$

$$|f\rangle = \sum_{i=0}^{\infty} c_i |i\rangle = \sum_{i=0}^{\infty} \langle i \mid f \rangle |i\rangle = \sum_{i=0}^{\infty} |i\rangle \langle i \mid f\rangle = \left(\sum_{i=0}^{\infty} |i\rangle \langle i| \right) |f\rangle$$

$$\sum_{i=0}^{\infty} |i\rangle \langle i| = 1$$

The **closure relation** ensures **completeness** of the basis set and vice versa

The bra for functions can be written in terms of an operator as

$$\langle f| = \int dx\, f^*(x) \circ$$

The Continuous Basis Set of Functions

$$\langle \phi_K \mid \phi_k \rangle = \delta(k - K)$$

orthonormality

$$\langle f \mid g \rangle = \int dx\, f^*(x) g(x)$$

inner product

$$|f\rangle = \int_a^b dk\, c_k |\phi_k\rangle$$

F has an integral expansion

$$f(x) = \int_a^b dk\, c_k\, \phi_k(x)$$

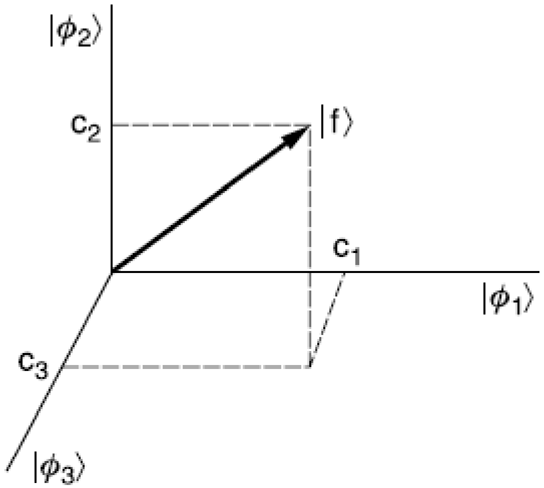


FIGURE 4.3.1
The function f projected onto the basis set of functions.

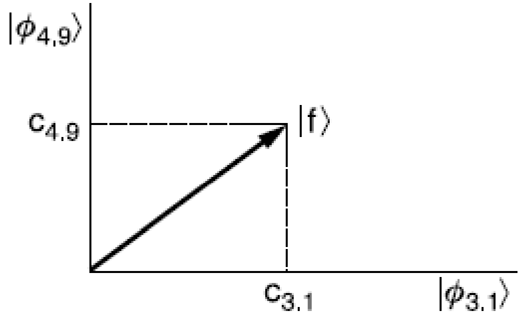


FIGURE 4.3.2
A function projected onto two of the many basis vectors.

Example 4.3.1

Is the set $\{1, x\}$ orthonormal on the interval $[-1, 1]$?

Note that the "1" and "x" represent functions and not coordinates. Therefore, define functions $f=1$ and $g=x$. These functions are orthogonal on the interval as can be seen

$$\langle f \mid g \rangle = \int_{-1}^{1} dx\, f^* g = \int_{-1}^{1} dx\, 1 \cdot x = 0$$

Neither function is normalized (unit length) since

$$\|f\|^2 = \langle f \mid f \rangle = \int_{-1}^{1} 1\, dx = 2 \quad \text{and} \quad \|g\|^2 = \langle g \mid g \rangle = \int_{-1}^{1} dx\, x^2 = \frac{2}{3}$$

In general, any function $h(x)$ can be normalized by redefining it as $h \to h/\|h\|$. An orthonormal set can be formed by dividing each function by its length. The orthonormal set is

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x \right\}$$

The Sine Basis Set

$$B_s = \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3 \ldots \right\} = \{\psi_n(x) : \; n = 1, 2, 3 \ldots\}$$

$$x \in (0, L)$$

Hilbert space can be expanded every 2L

$$|f\rangle = \sum_{m=1}^{\infty} c_m |\psi_m\rangle \quad \text{or} \quad f(x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

The expansion coefficients are found by

$$\langle \psi_n \mid f \rangle = \langle \psi_n | \left\{ \sum_m c_m |\psi_m\rangle \right\} = \sum_m c_m \langle \psi_n \mid \psi_m \rangle = c_n$$

$$c_n = \langle \psi_n \mid f \rangle = \left\langle \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \middle| f(x) \right\rangle = \sqrt{\frac{2}{L}} \int_0^L dx\, f(x) \sin\left(\frac{n\pi x}{L}\right)$$

By: Dr. Jabbari

The Cosine Basis Set

$$B_c = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right), \ \dots \ \text{for } n = 1, 2, 3 \dots \right\} = \{\phi_0, \phi_1, \ \dots\}$$

$$|f\rangle = \sum_{n=0}^{\infty} c_n |\phi_n\rangle$$

$$f(x) = \frac{c_0}{\sqrt{L}} + \sum c_n \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right)$$

$$c_0 = \langle \phi_0 \mid f \rangle = \left\langle \frac{1}{\sqrt{L}} \middle| f(x) \right\rangle = \frac{1}{\sqrt{L}} \int_0^L dx\, f(x)$$

$$c_n = \langle \phi_n \mid f \rangle = \left\langle \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) \middle| f(x) \right\rangle = \sqrt{\frac{2}{L}} \int_0^L dx\, f(x) \cos\left(\frac{n\pi x}{L}\right)$$

The Fourier Series Basis Set

interval (-L, L)

$$B = \left\{ \frac{1}{\sqrt{2L}} \exp\left(i\frac{n\pi x}{L}\right) \quad n = 0, \pm 1, \pm 2 \ldots \right\}$$

orthonormality relation

$$\left\langle \frac{1}{\sqrt{2L}} \exp\left(i\frac{n\pi x}{L}\right) \middle| \frac{1}{\sqrt{2L}} \exp\left(i\frac{m\pi x}{L}\right) \right\rangle = \delta_{nm}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{D_n}{\sqrt{2L}} \exp\left(i\frac{n\pi x}{L}\right)$$

The coefficients D_n can be complex.

The wave is required to repeat itself every length L instead of 2L given above.

$$B = \left\{ \frac{1}{\sqrt{L}} \exp\left(i\frac{2n\pi x}{L}\right) \quad n = 0, \pm 1, \pm 2 \quad \ldots \right\}$$

For three dimensions

$$B = \left\{ \frac{1}{\sqrt{V}} \exp\left(i\vec{k} \cdot \vec{r}\right) \right\}$$

$$V = L_x L_y L_z \quad \text{and} \quad k_x = (2\pi m/L_x), \ k_y = (2\pi n/L_y), \ k_z = (2\pi p/L_z)$$

The Fourier Transform

$$\left\{\frac{e^{ikx}}{\sqrt{2\pi}}\right\}$$

$$f(x) = \int_{-\infty}^{\infty} dk\ \alpha(k)\frac{e^{ikx}}{\sqrt{2\pi}}$$

$$\left\{|k\rangle = |\phi_k\rangle = \left|\frac{1}{\sqrt{2\pi}}e^{ik0}\right\rangle \quad \rightarrow \quad \phi_k(x) = \langle x \mid k\rangle = \frac{1}{\sqrt{2\pi}}\exp(ikx)\right\}$$

inner product

$$\boxed{\langle K \mid k\rangle = \int_{-\infty}^{\infty} dx\frac{e^{-iKx}}{\sqrt{2\pi}}\frac{e^{ikx}}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} dx\frac{e^{i(k-K)x}}{2\pi} = \delta(k-K)}$$

The closure relation

$$\boxed{\hat{1} = \int_{-\infty}^{\infty} |k\rangle dk\langle k|}$$

Summary of Results

	Euclidean Vectors	Functions-Discrete Basis	Functions-Continuous Basis
Basis	$\{\lvert n\rangle : n = 1,2,3\ldots\} \sim \{\tilde{x}, \tilde{y}, \tilde{z}\ldots\}$ $n = \text{Integer}$	$\{\lvert n\rangle = \lvert u_n\rangle \tilde{u}_n(x)\}$ $n = \text{Integer}$	$\{\lvert k\rangle = \lvert \phi_k\rangle \tilde{\phi}_k(x)\}$ $k = \text{Real}$
Projector	$\langle w\rvert = \vec{w}\cdot$	$\langle f\rvert = \int dx\, f^*(x) \circ$	$\langle f\rvert = \int dx\, f^*(x) \circ$
Orthonormality	$\langle m \mid n\rangle = \delta_{m,n}$	$\langle u_m \mid u_n\rangle = \delta_{mn}$	$\langle \phi_K \mid \phi_k\rangle = \delta(k - K)$
Complete	$\lvert v\rangle = \sum_n c_n \lvert n\rangle$	$\lvert f\rangle = \sum_n c_n \lvert u_n\rangle$ $f(x) = \sum_n c_n u_n(x)$	$\lvert f\rangle = \int dk\, c_k\, \lvert \phi_k\rangle$ $f(x) = \int dx\, c_k\, \phi_k(x)$
Components	$c_n = \langle n \mid v\rangle$	$c_n = \langle u_n \mid f\rangle$	$c_k = \langle \phi_k \mid f\rangle$
Inner Product	$\langle v \mid w\rangle = \sum_n v_n^* w_n$	$\langle f \mid g\rangle = \int dx\, f^*(x)\, g(x)$	$\langle f \mid g\rangle = \int dx\, f^*(x)\, g(x)$
Closure	$\sum_n \lvert n\rangle\langle n\rvert = \hat{1}$	$\Sigma_n \lvert u_n\rangle\langle u_n\rvert = \hat{1}$ $\delta(x - x') = \sum_n u_n^*(x')\, u_n(x)$	$\int dk\, \lvert \phi_k\rangle\langle \phi_k\rvert = \hat{1}$ $\delta(x - x') = \int dk\, \phi_k^*(x')\, \phi_k(x)$