دانشگاه آزاد
مرودشت

Boundary Conditions for the Electric and Magnetic Fields

جهت حل معادلات ماکسول و معادلات مشتقات جزیی نیاز مبرم به شرایط مرزی می باشد

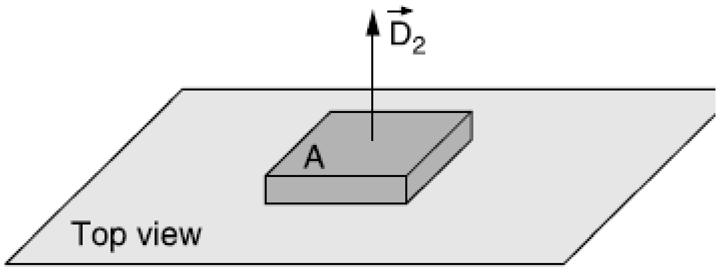
By: Dr. Jabbari

Electric Field Perpendicular to an Interface

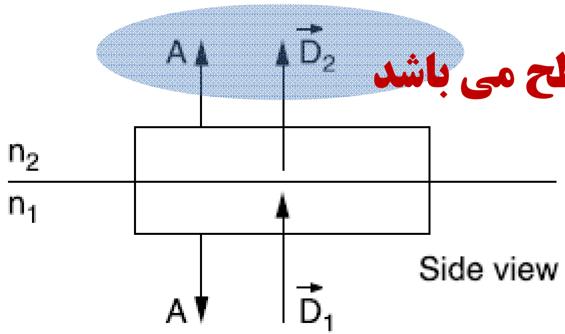
Case 1 No Free Surface Charge

$$n_1 = \sqrt{\varepsilon_1/\varepsilon_o} \text{ and } n_2 = \sqrt{\varepsilon_2/\varepsilon_o}$$

$$\nabla \cdot \vec{\mathscr{D}} = 0 \quad \Rightarrow \quad \int_V \nabla \cdot \vec{\mathscr{D}}\, dV = 0 \quad \rightarrow \quad \int_{A_T} \vec{\mathscr{D}} \cdot d\vec{a} = 0$$



$$0 = \int_{A_T} \vec{\mathscr{D}} \cdot d\vec{a} = \int_{A_{top}} \mathscr{D}_2\, da - \int_{A_{bottom}} \mathscr{D}_1\, da$$



جابجایی عمود بر سطح می باشد

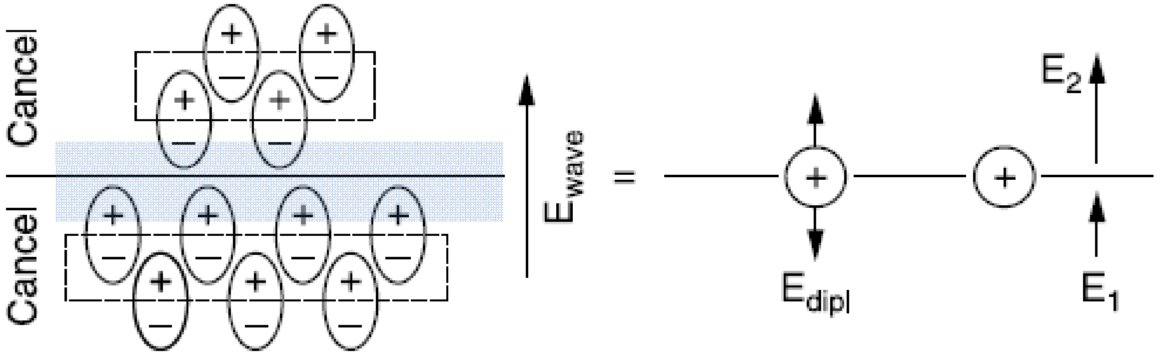
$$\vec{\mathscr{D}}_2 = \vec{\mathscr{D}}_1$$

$$\vec{\mathscr{D}} = \varepsilon \vec{\mathscr{E}}$$

جابجایی باید در مرز پیوسته باشد اما
میدان الکتریکی نمی تواند پیوسته باشد

$$\mathscr{E}_2 = \frac{\varepsilon_1}{\varepsilon_2} \mathscr{E}_1 = \left(\frac{n_1}{n_2}\right)^2 \mathscr{E}_1$$

tend not
to cancel
In the surface



$$\left(\mathscr{E}_{\text{wave}} + \mathscr{E}_{\text{dipole}}\right)_{\text{bottom}} = \mathscr{E}_1 \quad \text{and} \quad \left(\mathscr{E}_{\text{wave}} + \mathscr{E}_{\text{dipole}}\right)_{\text{top}} = \mathscr{E}_2$$

In the region where the dipole field points downward, the electric field decreases and where the dipole field points upward, the electric field increases. Therefore across the interface, there must be a discontinuity in the electric field

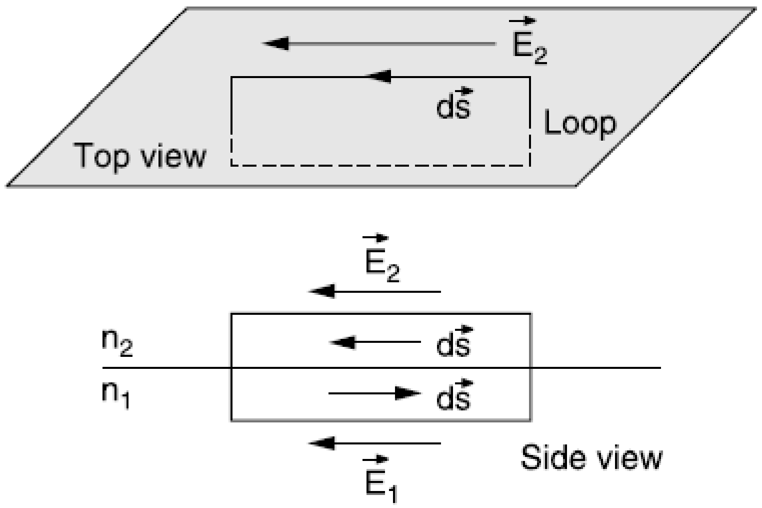
Case 2 With Free Surface Charges

$$\nabla \cdot \vec{\mathscr{D}} = \rho = \sigma_{\text{free}}\delta(z) \quad \rightarrow \quad \int_V \nabla \cdot \vec{\mathscr{D}}\, dV = \int_{A_T} \sigma_{\text{free}}\, da \quad \rightarrow \quad \int_{A_T} \vec{\mathscr{D}} \cdot d\vec{a} = \int_{A_T} \sigma_{\text{free}}\, da$$

Dirac delta function

$$\vec{\mathscr{D}}_2 = \vec{\mathscr{D}}_1 + \sigma_{\text{free}} \quad \rightarrow \quad \varepsilon_2 \vec{\mathscr{E}}_2 = \varepsilon_1 \vec{\mathscr{E}}_1 + \sigma_{\text{free}}$$

Electric Fields Parallel to the Surface



$$\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t}$$

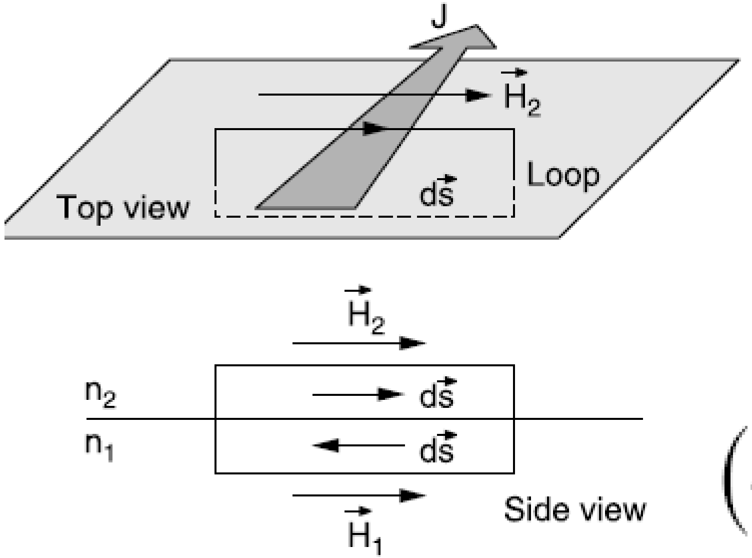
there aren't any free currents or changing magnetization

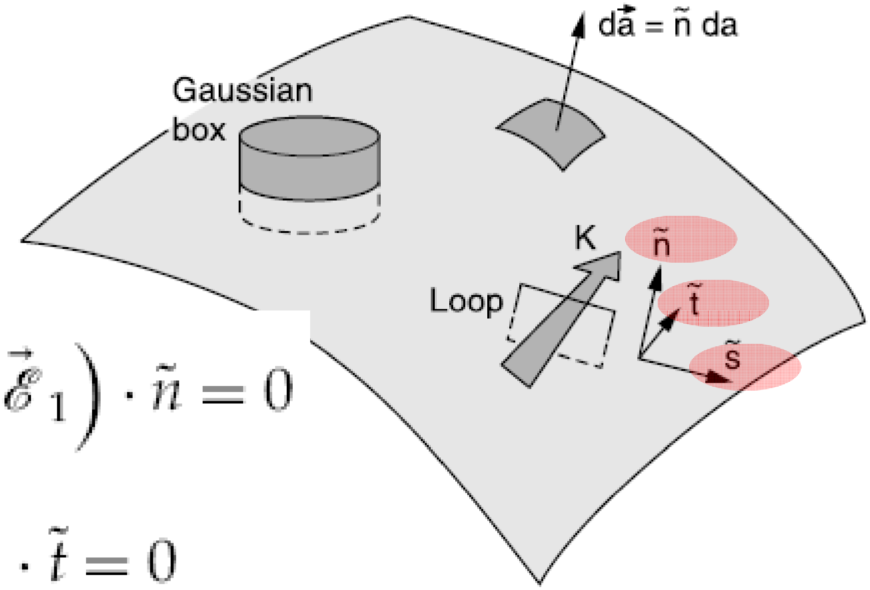
$$\nabla \times \vec{\mathscr{E}} = 0$$

$$\int_A \nabla \times \vec{\mathscr{E}} \cdot d\vec{a} = 0 \quad \rightarrow \quad \oint \vec{\mathscr{E}} \cdot d\vec{s} = 0$$

$$0 = \oint \vec{\mathscr{E}} \cdot d\vec{s} = \underbrace{\mathscr{E}_2 L}_{\text{Top}} + \underbrace{-\mathscr{E}_1 L}_{\text{Bottom}} \quad \rightarrow \mathscr{E}_2 = \mathscr{E}_1$$

با دنبال کردن همین روش برای میدان مغناطیس می توانیم میدانها را در شرایط مرزی بدست آوریم





$$\left(\varepsilon_2\vec{\mathscr{E}}_2 - \varepsilon_1\vec{\mathscr{E}}_1\right)\cdot\tilde{n} = 0$$

$$\left(\vec{\mathscr{E}}_2 - \vec{\mathscr{E}}_1\right)\cdot\tilde{t} = 0$$

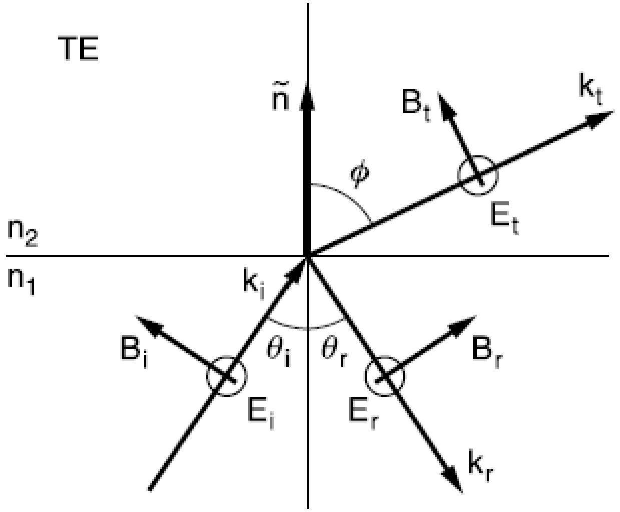
$$\left(\vec{\mathscr{H}}_2 - \vec{\mathscr{H}}_1\right)\cdot\tilde{s} = 0$$

$$\left(\vec{\mathscr{H}}_2 - \vec{\mathscr{H}}_1\right)\cdot\tilde{n} = 0$$

$$\left(\vec{\mathscr{E}}_2 - \vec{\mathscr{E}}_1\right)\cdot\tilde{s} = 0$$

$$\left(\vec{\mathscr{H}}_2 - \vec{\mathscr{H}}_1\right)\cdot\tilde{t} = 0$$

Law of Reflection, Snell's Law and the Reflectivity



$$\left(\varepsilon_2\vec{\mathscr{E}}_2-\varepsilon_1\vec{\mathscr{E}}_1\right)\cdot\tilde{n}=0$$

$$\left(\vec{\mathscr{E}}_2-\vec{\mathscr{E}}_1\right)\cdot\tilde{t}=0$$

$$\left(\vec{\mathscr{H}}_2-\vec{\mathscr{H}}_1\right)\cdot\tilde{n}=0$$

$$\left(\vec{\mathscr{H}}_2-\vec{\mathscr{H}}_1\right)\cdot\tilde{t}=0$$

$$\vec{\mathscr{E}}_1=\vec{\mathscr{E}}_i+\vec{\mathscr{E}}_r \qquad \vec{\mathscr{E}}_2=\vec{\mathscr{E}}_t$$

$$\vec{\mathscr{H}}_1=\vec{\mathscr{H}}_i+\vec{\mathscr{H}}_r \qquad \vec{\mathscr{H}}_2=\vec{\mathscr{H}}_r$$

$$\begin{cases}\left[\varepsilon_2\vec{\mathscr{E}}_t-\varepsilon_1\left(\vec{\mathscr{E}}_i+\vec{\mathscr{E}}_r\right)\right]\cdot\tilde{n}=0\\[2ex]\left[\vec{\mathscr{E}}_t-\left(\vec{\mathscr{E}}_i+\vec{\mathscr{E}}_r\right)\right]\times\tilde{n}=0\end{cases}$$

relation between the magnitude of the **magnetic** and **electric** fields in a **dielectric**

$$\mathscr{H} = \frac{\mathscr{E}}{\mu_o v_g}$$

Adding the direction

$$\left(\vec{k}_t \times \vec{\mathscr{E}}_t - \vec{k}_i \times \vec{\mathscr{E}}_i - \vec{k}_r \times \vec{\mathscr{E}}_r\right) \cdot \tilde{n} = 0$$

$$\left(\vec{k}_t \times \vec{\mathscr{E}}_t - \vec{k}_i \times \vec{\mathscr{E}}_i - \vec{k}_r \times \vec{\mathscr{E}}_r\right) \times \tilde{n} = 0$$

$$\vec{\mathscr{H}} = \tilde{k} \times \frac{\vec{\mathscr{E}}}{\mu_o v_g} = \frac{\vec{k}_n}{k_n} \times \frac{\vec{\mathscr{E}}}{\mu_o v_g} = \frac{\vec{k}_n}{k_o n} \times \frac{\vec{\mathscr{E}}}{\mu_o v_g} = \frac{\vec{k}_n \times \vec{\mathscr{E}}}{k_o \mu_o c}$$

$$\vec{\mathscr{H}} = \frac{\vec{k} \times \vec{\mathscr{E}}}{k_o \mu_o c}$$

Let's group the boundary conditions together to write

$$\left[\varepsilon_2\vec{\mathscr{E}}_t-\varepsilon_1\left(\vec{\mathscr{E}}_i+\vec{\mathscr{E}}_r\right)\right]\cdot\tilde{n}=0$$

$$\left[\vec{\mathscr{E}}_t-\left(\vec{\mathscr{E}}_i+\vec{\mathscr{E}}_r\right)\right]\times\tilde{n}=0$$

$$\left(\vec{k}_t\times\vec{\mathscr{E}}_t-\vec{k}_i\times\vec{\mathscr{E}}_i-\vec{k}_r\times\vec{\mathscr{E}}_r\right)\cdot\tilde{n}=0$$

$$\left(\vec{k}_t\times\vec{\mathscr{E}}_t-\vec{k}_i\times\vec{\mathscr{E}}_i-\vec{k}_r\times\vec{\mathscr{E}}_r\right)\times\tilde{n}=0$$

The Law of Reflection

we show that the angle of incidence equals the angle of reflection

$$\vec{\mathscr{E}} = \vec{\mathscr{E}}_o \, e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

$$\left[\varepsilon_2 \vec{\mathscr{E}}_t - \varepsilon_1\left(\vec{\mathscr{E}}_i + \vec{\mathscr{E}}_r\right)\right]\cdot\tilde{n} = 0 \quad \Rightarrow$$

$$\varepsilon_2 \vec{\mathscr{E}}_{ot}\cdot\tilde{n}\, e^{i\vec{k}_t\cdot\vec{r}-i\omega t} = \varepsilon_1 \vec{\mathscr{E}}_{oi}\cdot\tilde{n}\, e^{i\vec{k}_i\cdot\vec{r}-i\omega t} + \varepsilon_1 \vec{\mathscr{E}}_{or}\cdot\tilde{n}\, e^{i\vec{k}_r\cdot\vec{r}-i\omega t}$$

Finally the results are:

$$\theta_i = \theta_r \quad \text{(Law of Reflection)}$$

$$n_2 \sin\phi = n_1 \sin\theta_i \quad \text{(Snell's law)}$$

Fresnel Reflectivity and Transmissivity for TE Fields

از ۴ معادله شرایط مرزی ۲ معادله کارایی ندارد

برای انعکاس این دو معادله کارایی دارد

$$\left[\vec{\mathscr{E}}_t - \left(\vec{\mathscr{E}}_i + \vec{\mathscr{E}}_r\right)\right] \times \tilde{n} = 0$$

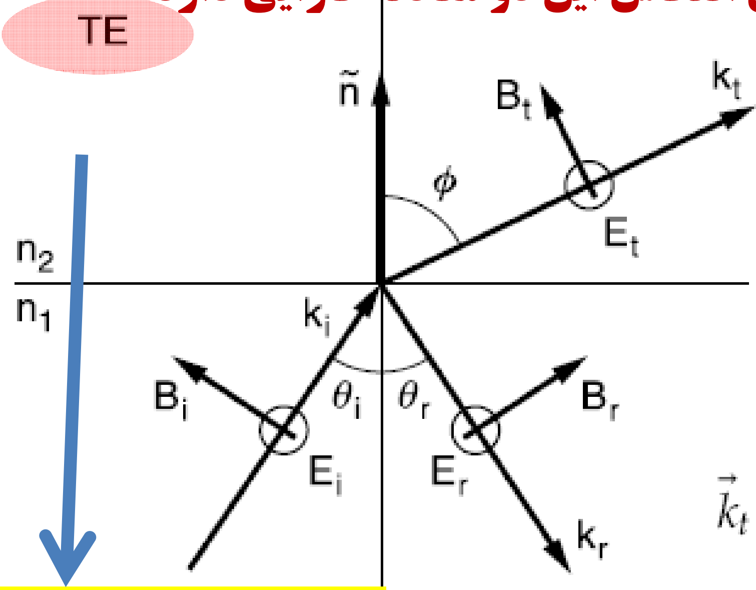
بدلیل عمود بودن

$$\mathscr{E}_t - (\mathscr{E}_i + \mathscr{E}_r) = 0$$

Eq.1

$$\left(\vec{k}_t \times \vec{\mathscr{E}}_t - \vec{k}_i \times \vec{\mathscr{E}}_i - \vec{k}_r \times \vec{\mathscr{E}}_r\right) \times \tilde{n} = 0$$

TE



The electric field is everywhere perpendicular to the wave vector

BAC–CAB

$$\vec{k}_t \times \vec{\mathscr{E}}_t \times \tilde{n} = \vec{\mathscr{E}}_t\left(\vec{k}_t \cdot \tilde{n}\right) - \tilde{n}\left(\vec{k}_t \cdot \vec{\mathscr{E}}_t\right) = \vec{\mathscr{E}}_t\left(\vec{k}_t \cdot \tilde{n}\right)$$

$$\vec{k}_t \times \vec{\mathscr{E}}_t \times \tilde{n} = \vec{\mathscr{E}}_t\left(\vec{k}_t \cdot \tilde{n}\right) = \vec{\mathscr{E}}_t k_t \cos\phi = \vec{\mathscr{E}}_t k_o n_2 \cos\phi$$

$$\mathscr{E}_t k_o n_2 \cos\phi - \mathscr{E}_i k_o n_1 \cos\theta + \mathscr{E}_r k_o n_1 \cos\theta = 0$$

Eq.2

Eq.1

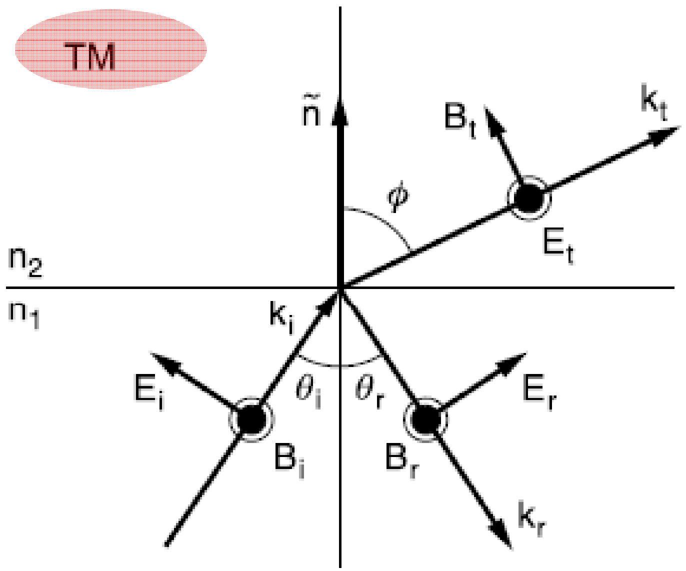
Eq.2

$$r_{TE} = \frac{\mathscr{E}_r}{\mathscr{E}_i} = \frac{n_1 \cos\theta - n_2 \cos\phi}{n_1 \cos\theta + n_2 \cos\phi}$$

$$n_2 \sin\phi = n_1 \sin\theta_i \quad \text{(Snell's law)}$$

$$t_{TE} = \frac{\mathscr{E}_t}{\mathscr{E}_i} = \frac{2n_1 \cos\theta}{n_1 \cos\theta + n_2 \cos\phi}$$

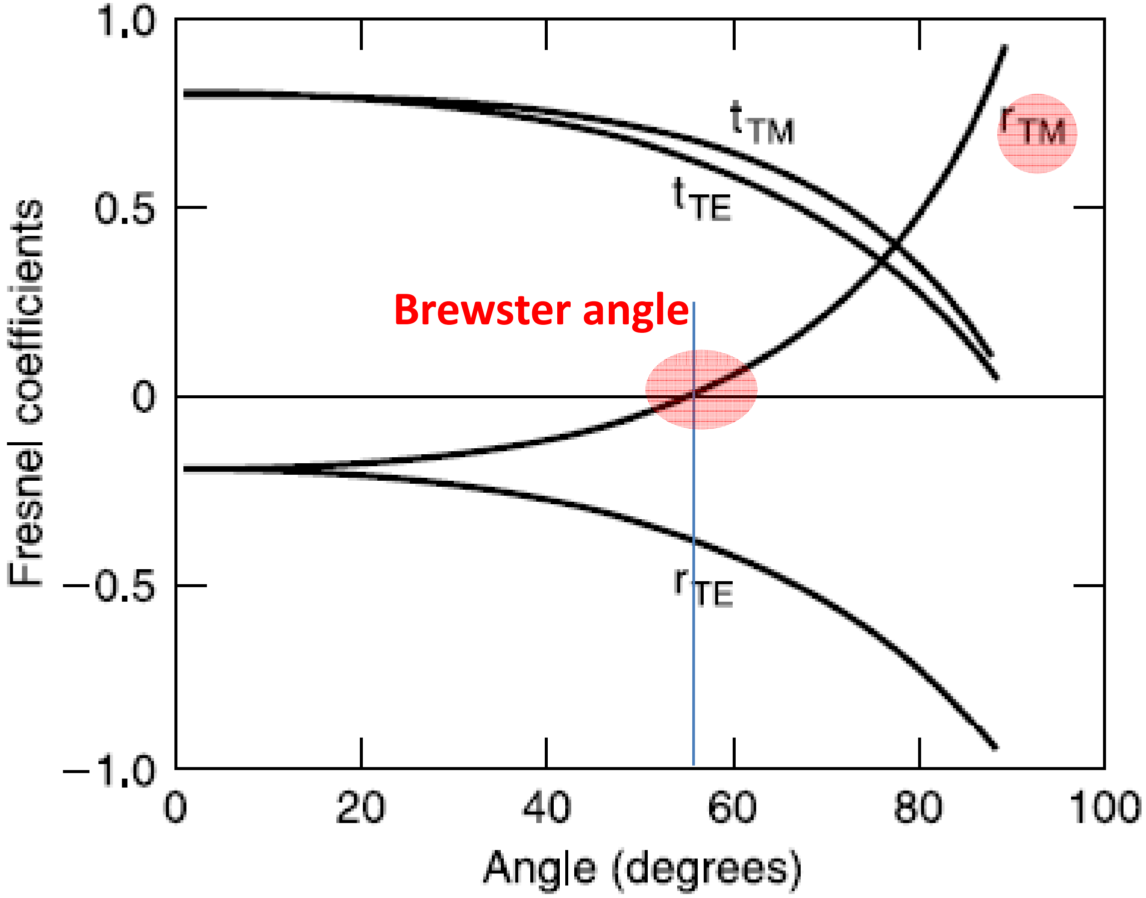
And for The TM Fields



$$r_{TM} = \frac{E_r}{E_i} = \frac{n_1 \cos\,\phi - n_2 \cos\,\theta}{n_2 \cos\,\theta + n_1 \cos\,\phi}$$

$$t_{TM} = \frac{E_t}{E_i} = \frac{n_1}{n_2}(1 + r_{TM})$$

The reflectivity and transmittivity versus the angle of incidence for n1=1 and n2=1.5



The Poynting Vector

The Poynting vector describes the energy flow from the material

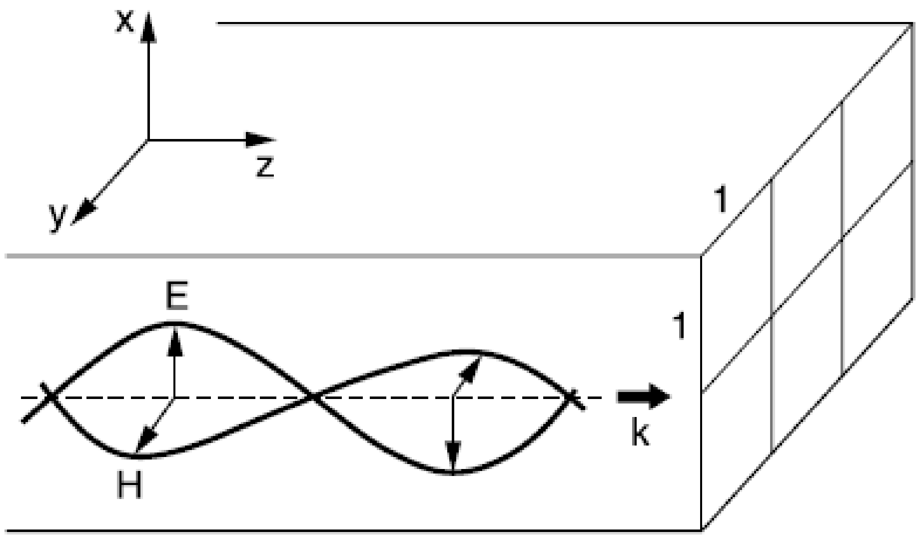
Introduction to Power Transport for Real Fields and Complex Fields

Now Real Fields

$$\vec{S} = \vec{E} \times \vec{H}$$

انرژی جاری یافته شامل میدان و پلاریزاسیون است

بردار پوینگتین توان جاری
شده در هر قسمت را بر
حسب زمان به ما می دهد



$$\begin{cases} \vec{\mathscr{E}}(z,t) = E_o \sin(k_o z - \omega t)\tilde{x} \\ \vec{\mathscr{H}}(z,t) = H_o \sin(k_o z - \omega t)\tilde{y} \end{cases}$$

$$\vec{\mathscr{S}} = \vec{\mathscr{E}} \times \vec{\mathscr{H}} = \tilde{z} E_o H_o \sin^2(k_o z - \omega t)$$

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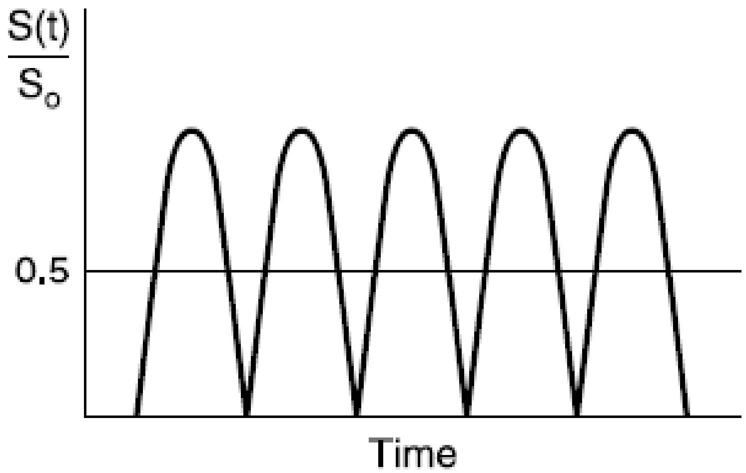

محاسبات ساده نشان می دهد که توان به صورت زیر در زمان حرکت می کند که مرتبه تغییر آن توانی از ۱٦ می باشد

The **instantaneous** power P(z, t) (**Watts through the surface**) depends on both the position z and time t

$$P(z,t) = \vec{S}\cdot\vec{A} = \vec{S}\cdot A\tilde{z}$$

$$P(z_o,t) = \vec{S}\cdot A\tilde{z} = AE_oH_o\,\mathrm{Sin}^2(k_oz_o - \omega t)$$

average power (averaged over a cycle)

$$\left\langle \vec{P}(z,t)\right\rangle_t = \frac{1}{T}\int_0^T dt\,\vec{P}(z,t) \qquad \left\langle P(z,t)\right\rangle_t = \tfrac{1}{2}AE_oH_o$$

$$T = 2\pi/\omega.$$

The Poynting vector can still depend on time, the amplitudes E_0 and H_0 **depend on time** with frequency **less than 100GHz**

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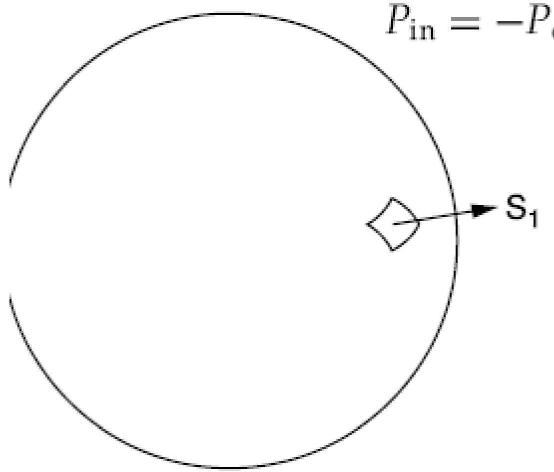
Power flows through a patch of area da.

$P_{\text{in}} = -P_{\text{out}}.$



The **total power** leaving the **surface** must be the **sum** of the power through each patch

$$P = \sum_{\text{surface}} \vec{S}(\vec{r}, t) \cdot d\vec{a}_{\vec{r}} = \int_{\text{surface}} \vec{S}(\vec{r}, t) \cdot d\vec{a}$$

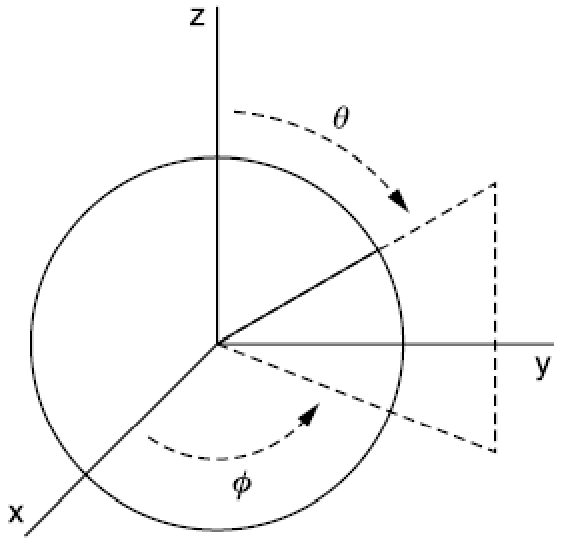
Example 3.5.1 $\vec{S} = \tilde{r}/r^2$

$$da = r^2 \sin\theta \, d\theta \, d\varphi$$

$$P = \int \vec{S} \cdot d\vec{a} = \int\int \frac{\tilde{r}}{R_o^2} \cdot \tilde{r} R_o^2 \sin\theta \, d\theta \, d\varphi = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\varphi = 4\pi$$

Power Transport and Energy Storage Mechanisms

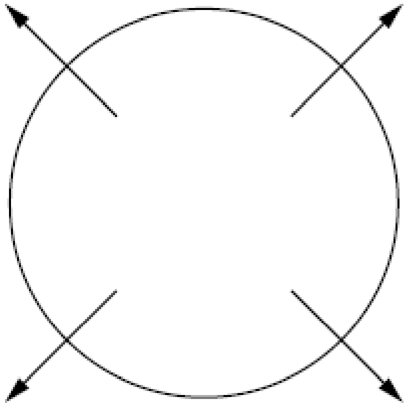
هدف: محاسبه چگونگی خارج شدن توان از یک حجم (البته در اصل باید یک توان وارد و سپس یک توان خارج شود.



روشهای متعددی برای ذخیره شدن انرژی وجود دارد:

First the electric and magnetic fields store energy
Second, electric and magnetic dipoles store energy
Third, the interaction between currents and fields can generate energy

We need to calculate the Poynting vector by using Maxwell's equations



divergence

$$\int_{\text{volume}} \nabla \cdot \vec{S}\, dV = \int_{\text{surface}} \vec{S} \cdot d\vec{a} = P_{\text{out}}$$

$$\nabla \cdot \vec{S} = \nabla \cdot \left(\vec{\mathscr{E}} \times \vec{\mathscr{H}}\right) = \vec{\mathscr{H}} \cdot \nabla \times \vec{\mathscr{E}} - \vec{\mathscr{E}} \cdot \nabla \times \vec{\mathscr{H}}$$

$$\begin{Bmatrix} \nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \dfrac{\partial \vec{\mathscr{D}}}{\partial t} \\ \nabla \times \vec{\mathscr{E}} = -\dfrac{\partial \vec{\mathscr{B}}}{\partial t} \end{Bmatrix} \quad \text{with} \quad \begin{Bmatrix} \vec{\mathscr{D}} = \varepsilon_0 \vec{\mathscr{E}} + \vec{\mathscr{P}} \\ \vec{\mathscr{B}} = \mu_0 \vec{\mathscr{H}} + \mu_0 \vec{\mathscr{M}} \end{Bmatrix}$$

از قبل
داریم

$$\nabla \cdot \vec{S} = \vec{\mathscr{H}} \cdot \left(\nabla \times \vec{\mathscr{E}}\right) - \vec{\mathscr{E}} \cdot \nabla \times \vec{\mathscr{H}} = \vec{\mathscr{H}} \cdot \left(-\frac{\partial \vec{\mathscr{B}}}{\partial t}\right) - \vec{\mathscr{E}} \cdot \left(\vec{J} + \frac{\partial \vec{\mathscr{D}}}{\partial t}\right)$$

$$\nabla \cdot \vec{S} = -\vec{\mathscr{H}} \cdot \frac{\partial}{\partial t}\left(\mu_0 \vec{\mathscr{H}} + \mu_0 \vec{\mathscr{M}}\right) - \vec{\mathscr{E}} \cdot \vec{\mathscr{J}} - \vec{\mathscr{E}} \cdot \frac{\partial}{\partial t}\left(\varepsilon_0 \vec{\mathscr{E}} + \vec{\mathscr{P}}\right)$$

Multiply by –1 and observe that rewriting the derivative as

$$\vec{\mathscr{H}} \cdot \frac{\partial \vec{\mathscr{H}}}{\partial t} = \frac{\partial}{\partial t} \frac{\vec{\mathscr{H}} \cdot \vec{\mathscr{H}}}{2} \quad \text{and} \quad \vec{\mathscr{E}} \cdot \frac{\partial \vec{\mathscr{E}}}{\partial t} = \frac{\partial}{\partial t} \frac{\vec{\mathscr{E}} \cdot \vec{\mathscr{E}}}{2}$$

$$-\nabla\cdot\vec{S}=\frac{\partial}{\partial t}\left(\frac{\mu_o}{2}\mathscr{H}^2+\frac{\varepsilon_o}{2}\mathscr{E}^2\right)+\vec{\mathscr{E}}\cdot\vec{\mathscr{J}}+\mu_o\vec{\mathscr{H}}\cdot\frac{\partial\vec{\mathscr{M}}}{\partial t}+\vec{\mathscr{E}}\cdot\frac{\partial\vec{P}}{\partial t}$$

$$P_{\text{out}}=\int\nabla\cdot\vec{S}\,dV \qquad\qquad P_{\text{in}}=-\int\nabla\cdot\vec{S}\,dV$$

Substituting for the divergence of the Poynting vector gives

$$P_{\text{in}}=\underbrace{\int\vec{\mathscr{E}}\cdot\vec{\mathscr{J}}\,dV}_{\text{damping}}+\underbrace{\frac{\partial}{\partial t}\int dV\left(\frac{\mu_o}{2}\mathscr{H}^2+\frac{\varepsilon_o}{2}\mathscr{E}^2\right)}_{\text{field energy}}+\underbrace{\int dV\,\vec{\mathscr{E}}\cdot\frac{\partial\vec{\mathscr{P}}}{\partial t}}_{\text{polarization energy}}+\underbrace{\int dV\,\mu_o\vec{\mathscr{H}}\cdot\frac{\partial\vec{\mathscr{M}}}{\partial t}}_{\text{magnetization energy}}$$

Poynting Vector for Complex Fields

$$\vec{S} = \frac{1}{2}\vec{\mathscr{E}} \times \vec{\mathscr{H}}^*$$

$$\text{plane wave} \quad \tilde{\mathscr{E}} \sim e^{ikz-i\omega t} \Longrightarrow \tilde{\mathscr{E}}\tilde{\mathscr{E}}^* = |\tilde{\mathscr{E}}|^2.$$

Example 3.5.2:
For the plane waves given in Section 3.5.1, specifically

$$\vec{\mathscr{E}}(z,t) = E_o \sin(k_o z - \omega t)\tilde{x} \qquad \vec{\mathscr{H}}(z,t) = H_o \sin(k_o z - \omega t)\tilde{y}$$

$\vec{S}$ **?**

complex notation

$$\vec{\mathscr{E}} = E_o e^{ikz-i\omega t} e^{i\varphi}\tilde{x} \quad \text{and} \quad \vec{\mathscr{H}} = H_o e^{ikz-i\omega t} e^{i\varphi}\tilde{y}$$

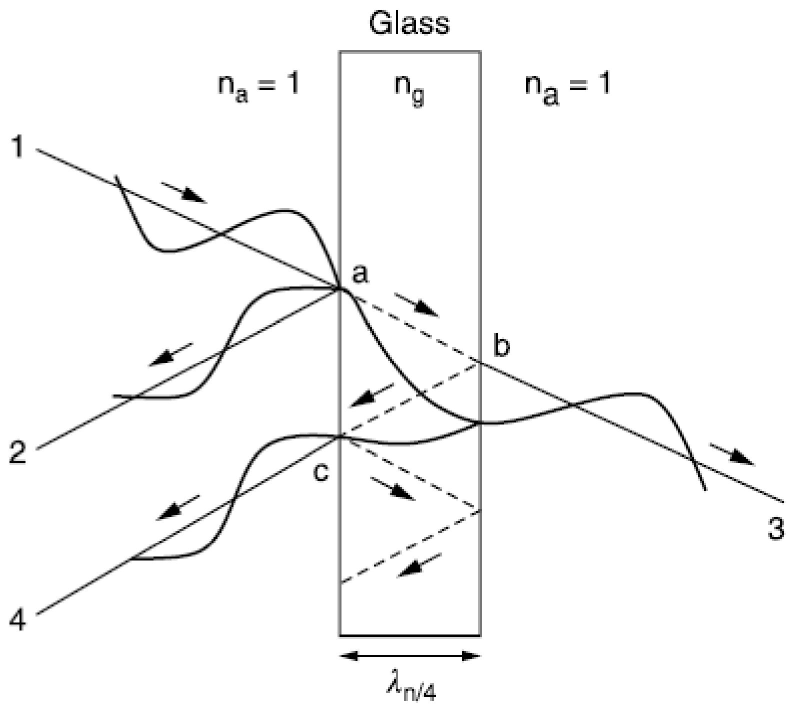
$$\vec{S} = \tfrac{1}{2}\vec{\mathscr{E}} \times \vec{\mathscr{H}}^* = \tfrac{1}{2}\tilde{z}\, E_o\, e^{ikz-i\omega t}\, e^{i\varphi} H_o^*\, e^{-ikz+i\omega t}\, e^{-i\varphi} = \tilde{z}\,\tfrac{1}{2}E_o H_o^*$$

$$\langle\vec{S}\rangle = \frac{1}{T}\int_0^T dt\, \vec{\mathscr{E}} \times \vec{\mathscr{H}} = \frac{1}{T}\int_0^T dt\, \tilde{z}\, E_o \sin(kz-\omega t)\, H_o \sin(kz-\omega t) = \tfrac{1}{2}\tilde{z}\, E_o H_o$$

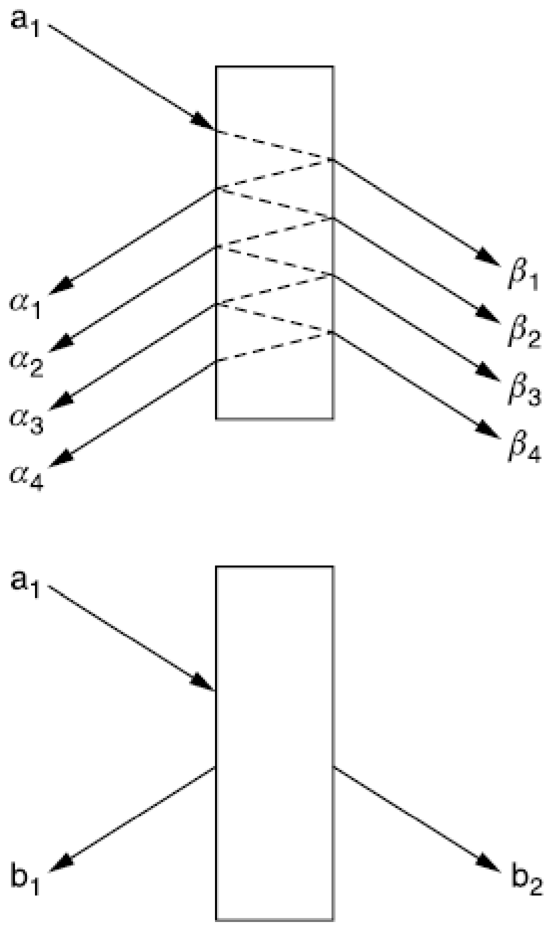
3.6 Electromagnetic Scattering and Transfer Matrix Theory

Where we have the two parallel mirrors like Fp and Vcele is the suitable for use the TMM

Introduction to Scattering Theory



The **phase** can be affected by the t**hickness** of the **plate**, the type of **material,** and the **reflection and transmission coefficients** at an interface.



where **S$_{ij}$** symbolizes the scattering matrix and NOT the Poynting vector. **The scattering matrix describes the particular optical element**. **Most importantly, the output is linearly related to the input**.

$$b_1 = \sum_i \alpha_i$$

$$b_2 = \sum_i \beta_i$$

$$b_1 = S_{11} a_1$$

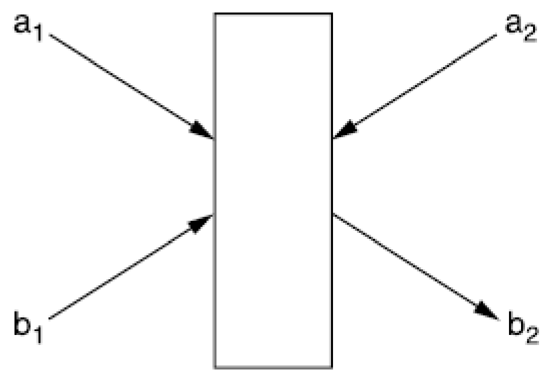
$$b_2 = S_{21} a_1$$

FIGURE 3.6.2
The reflect α_i and transmitted β_i amplitudes ad together to produce b_1 and b_2, respectively.

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underline{S} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$



The Power-Amplitudes

$$\begin{cases} \vec{\mathscr{E}} = \tilde{x}\, E_o\, u(x,y) \exp(ik_n z - i\omega t) \\ \vec{\mathscr{H}} = \tilde{y}\, H_o\, u(x,y) \exp(ik_n z - i\omega t) \end{cases}$$

The function $u(x, y)$ can be normalized such that

$$\int dx\, dy\, |u(x,y)|^2 = 1$$

Often we simply take ''u=constant'' for convenience.
The **phasor representations** for the electric and magnetic fields become

$$E(z) = E_o\, u \exp(ik_n z)$$

$$H(z) = H_o\, u \exp(ik_n z)$$

If the ''power amplitudes'' =a(z) then the total power will be: $P = aa^*$

To find the amplitude ''a,''

$$\vec{S} = \tfrac{1}{2}\left(\vec{\mathscr{E}} \times \vec{\mathscr{H}}^*\right) \xrightarrow{H = \frac{E}{\mu_o v} \quad \text{and} \quad v = c/n}$$

$$\varepsilon = n^2 \varepsilon_o$$

$$\vec{S} = \tfrac{1}{2} \tilde{z}\, EH^* |u|^2 = \tfrac{1}{2} \tilde{z}\, E \frac{E^*}{\mu_o v} |u|^2 = \frac{|E|^2 |u|^2}{2\mu_o v}\, \tilde{z} = \frac{\varepsilon v}{2} |E|^2 |u|^2 \tilde{z}$$

$$P = a(z)a^*(z) = \int dxdy\, \tilde{z}\cdot\vec{S} = \frac{\varepsilon v}{2}|E|^2 \int dx\, dy\, |u|^2 = \frac{\varepsilon v}{2}|E|^2$$

$$v_g = c\frac{1}{n} = \frac{1}{\sqrt{\varepsilon_o \mu_o}}\sqrt{\frac{\varepsilon_o}{\varepsilon}} = \frac{1}{\sqrt{\varepsilon \mu_o}}$$

$$P = a(z)a^*(z) = \frac{\varepsilon v}{2}|E|^2 = \frac{\sqrt{\varepsilon}}{2\sqrt{\mu_o}}|E|^2 = \frac{1}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}\sqrt{\frac{\varepsilon}{\varepsilon_o}}|E|^2 = \frac{\varepsilon_o c}{2}\, nE(z)E^*(z)$$

$$a(z) = \sqrt{\frac{\varepsilon_o c}{2} n}\; E_o e^{ik_n z}$$

to account for changes in the phase of the wave due to propagation

does not contribute when the power is calculated $P = aa^*$

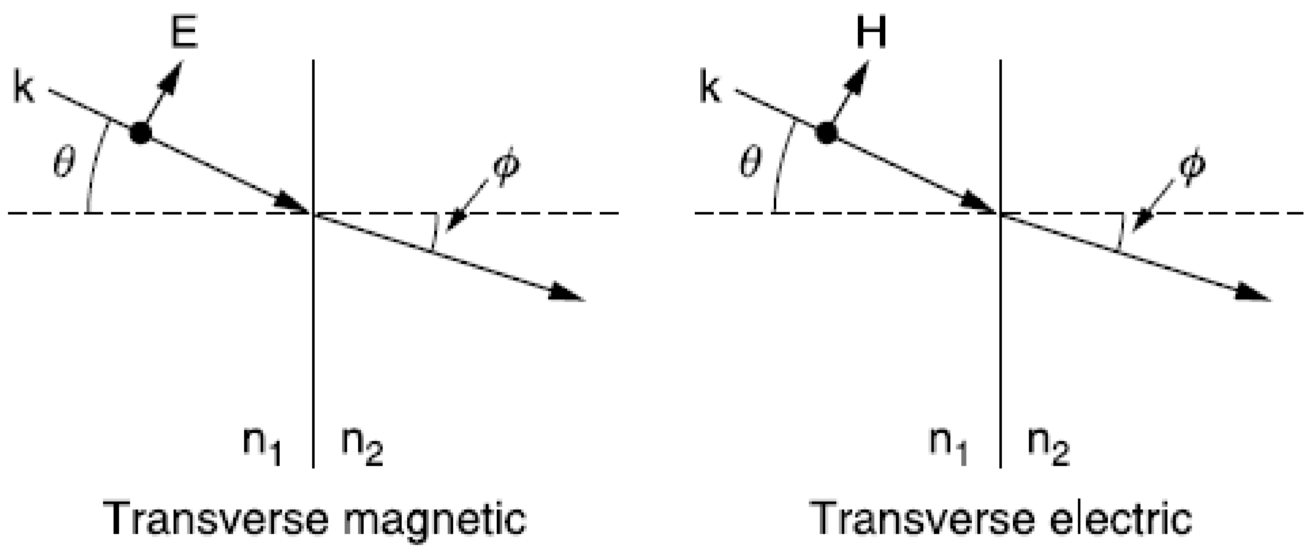
By: Dr. Jabbari

Reflection and Transmission Coefficients

$$a(z) = \sqrt{\frac{\varepsilon_o c}{2}}\, n\, E_o e^{ik_n z}$$

transverse magnetic **(TM)** wave has the magnetic field parallel to the interface (transverse to the plane of incidence)

while a transverse electric **(TE)** wave has the electric field parallel to the interface (transverse to the plane of incidence).



The reflection coefficients can be written as follows

$$r = \frac{n_1 \cos\theta - n_2 \cos\phi}{n_1 \cos\theta + n_2 \cos\phi} \qquad \text{TE}$$

$$r = \frac{-n_2 \cos\theta + n_1 \cos\phi}{n_2 \cos\theta + n_1 \cos\phi} \qquad \text{TM}$$

$\theta = 0$ and $\phi = 0$ $\Longrightarrow$ $$r = \frac{n_1 - n_2}{n_1 + n_2}$$

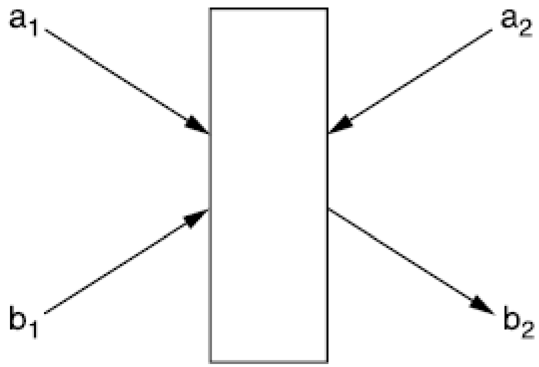
$$t = \frac{2\cos\theta \sin\phi}{\sin(\theta + \phi)} \qquad \text{TE}$$

$$t = \frac{2\cos\theta \sin\phi}{\sin(\theta + \phi)\cos(\theta - \phi)} \qquad \text{TM}$$

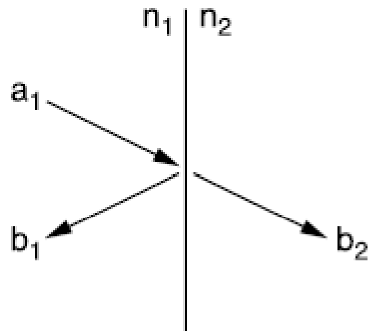
$$\sin(\theta + \phi) = \sin\theta\,\cos\phi - \sin\phi\,\cos\theta$$

$(\theta = 0 = \phi)$ $\Longrightarrow$ $$t_{1\to 2} = \frac{2n_1}{n_1 + n_2} \quad \text{(Normal Incidence)}$$

Question: How do the Fresnel coefficients relate to reflection and transmission coefficients?



$$r_{pa} = \frac{b_1}{a_1} = \left.\frac{\sqrt{n_1}E_{b1}e^{-ik_n z}}{\sqrt{n_1}E_{a1}e^{ik_n z}}\right|_{z=0} = \frac{E_{b1}}{E_{a1}} = r$$



$$\tau_{1\to 2} = \frac{b_2}{a_1} = \left.\frac{\sqrt{n_2}E_{b2}e^{ik_{n2}z}}{\sqrt{n_1}E_{a1}e^{ik_{n1}z}}\right|_{z=0} = \sqrt{\frac{n_2}{n_1}}\frac{E_{b2}}{E_{a1}} = \sqrt{\frac{n_2}{n_1}}\, t_{1\to 2}$$

$$R = \frac{P_{\mathrm{refl}}}{P_{\mathrm{inc}}} = \frac{b_1 b_1^*}{a_1 a_1^*} = \left(\frac{b_1}{a_1}\right)\left(\frac{b_1}{a_1}\right)^* = rr^* = |r|^2$$

$$T = \frac{P_{\text{trans}}}{P_{\text{inc}}} = \frac{b_2 b_2^*}{a_1 a_1^*} = \left(\frac{b_2}{a_1}\right)\left(\frac{b_2}{a_1}\right)^* = \tau_{1\to 2}\tau_{1\to 2}^* = |\tau_{1\to 2}|^2$$

Finally, we can write energy conservation

$$P_{\text{inc}} = P_{\text{refl}} + P_{\text{trans}}$$

$$\Downarrow$$

$$P_{\text{inc}} = RP_{\text{inc}} + TP_{\text{inc}}$$

$$\Downarrow$$

$$R + T = 1 \Rightarrow R = |r|^2 \qquad T = \frac{n_2}{n_1}|t_{1\to 2}|^2 = |t|^2$$

For normal incidence, the TE and TM power-amplitude

$$r = -\frac{n_2 - n_1}{n_2 + n_1} \quad \tau_{1\to 2} = \sqrt{\frac{n_2}{n_1}}\, t_{1\to 2} = \frac{2\sqrt{n_1 n_2}}{n_1 + n_2}$$

$$1 = R + T = |r|^2 + |\tau|^2$$

&

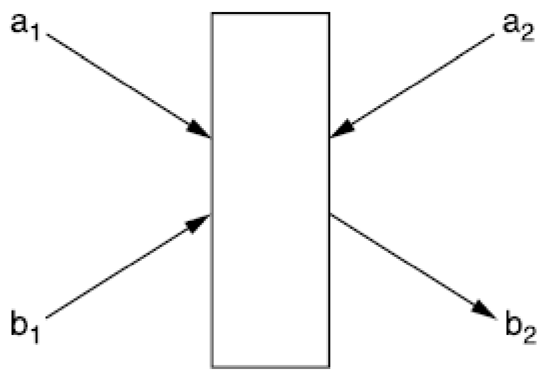
$$\tau^2 = t_{1\to 2} t_{2\to 1}$$

Scattering Matrices

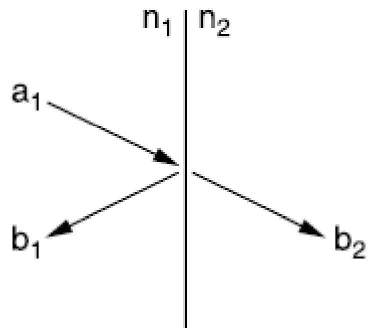
"Operators" represent the operations that the optical element performs on optical beams

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underline{S}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11}a_1 + S_{12}a_2 \\ S_{21}a_1 + S_{22}a_2 \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11}a_1 + S_{12}a_2 \\ S_{21}a_1 + S_{22}a_2 \end{pmatrix}$$



The simplest optical element is the interface



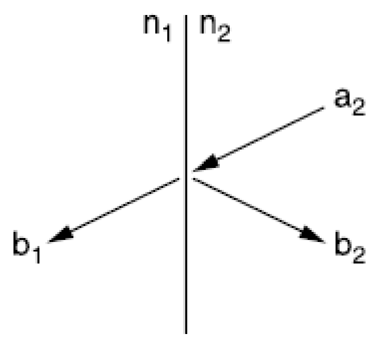
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underline{S} \begin{pmatrix} a_1 \\ 0 \end{pmatrix} = \begin{pmatrix} S_{11} a_1 \\ S_{21} a_1 \end{pmatrix}$$

$S_{11} = b_1/a_1$ $\qquad S_{11} = -r \qquad 0 \leq |r| \leq 1$

$S_{21} = b_2/a_1$ $\qquad S_{21} = \tau_{1 \to 2}$

phase change of approximately 180 for a wave reflecting from a medium with larger refractive index

For example, suppose a_1 =0 but that a_2 ‡ 0. The **scattering matrix** becomes:

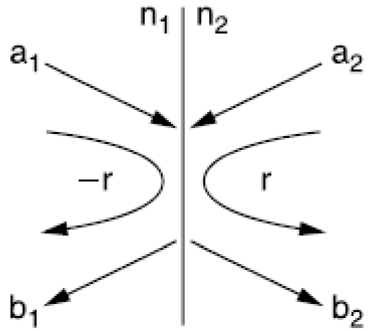


$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{12} a_2 \\ S_{22} a_2 \end{pmatrix}$$

with $S_{21} = \tau$ and $S_{22} = +r$.

superposition

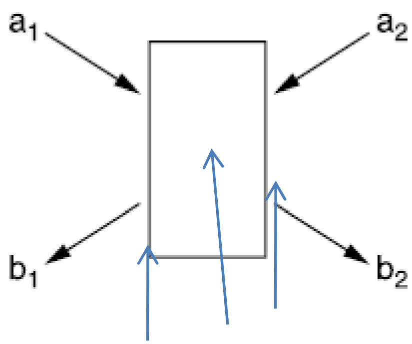
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underline{S}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -r & \tau_{2\to 1} \\ \tau_{1\to 2} & r \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$



Note the convention of ‘‘–r’’ and ‘‘r’’ in the figure makes **r>0** for **n1<n2.**
If in reality n1>n2, then the sign of ‘‘r’’ will be reversed; either way, the equations work out **ok.**

finite thickness with two interfaces

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underline{S}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -r_{\text{eff}} & t_{\text{eff}} \\ t_{\text{eff}} & -r_{\text{eff}} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$



$$-r_{\text{eff}} = b_1 / a_1$$

effective reflection and transmission coefficients

three different scattering matrices

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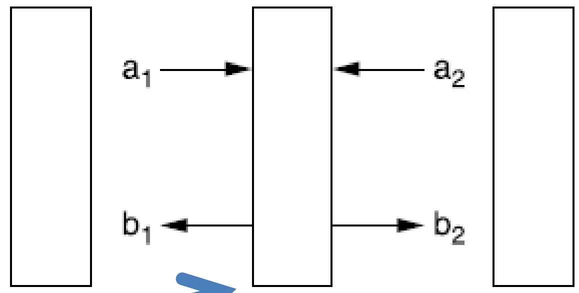
مرودشت

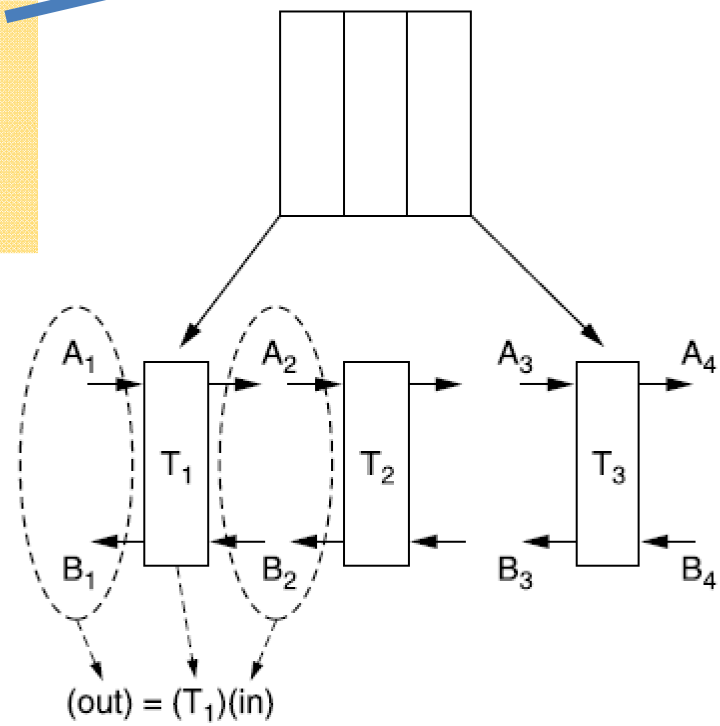
Scattering Matrix versus Transfer Matrix

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \underline{T}_1 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \underline{T}_1 \underline{T}_2 \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$$



The **scattering matrix** describes the reflected and transmitted amplitudes at an interface

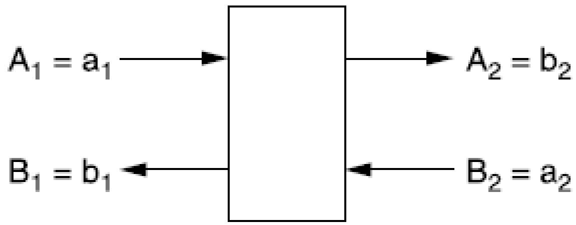
The scattering matrix **does not provide** a convenient formulation for solving **more complicated** problems
So this is where the transfer matrix comes to play a great role



Start to find a relation between the scattering and transfer matrices:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underline{S} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

On the other hand:



$$B_1 = S_{11}A_1 + S_{12}B_2$$
$$A_2 = S_{21}A_1 + S_{22}B_2$$

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

$$A_1 = \frac{1}{S_{21}}A_2 - \frac{S_{22}}{S_{21}}B_2$$

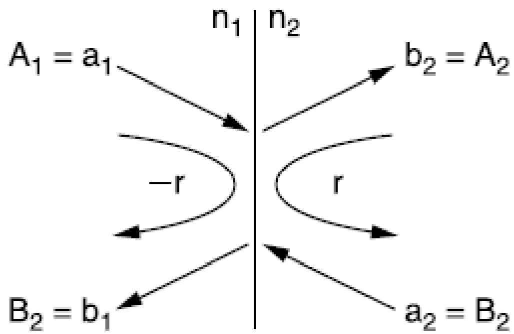
$$B_1 = S_{11}A_1 + S_{12}B_2 = \frac{S_{11}}{S_{21}}A_2 - \frac{S_{11}S_{22} - S_{12}S_{21}}{S_{21}}B_2$$

$$T = \frac{1}{S_{21}}\begin{pmatrix} 1 & -S_{22} \\ S_{11} & -\mathrm{Det}(S) \end{pmatrix} \quad \& \quad S = \frac{1}{T_{11}}\begin{pmatrix} T_{21} & \mathrm{Det}\, T \\ 1 & -T_{12} \end{pmatrix}$$

Example 3.6.1 The Simple Interface

$$\begin{cases} A_i(z) = A_{oi}\, e^{ik_i z} \\ B_i(z) = B_{oi}\, e^{-ik_i z} \end{cases}$$

$$k_i = k_o n_i (n_i \text{ is real})$$



$$\underline{S} = \begin{pmatrix} -r & \tau_{2\to1} \\ \tau_{1\to2} & r \end{pmatrix}$$

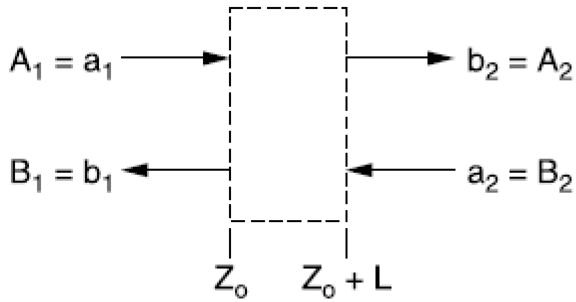
$$\begin{pmatrix} A_1(z) \\ B_1(z) \end{pmatrix}_{z=0} = T \begin{pmatrix} A_2(z) \\ B_2(z) \end{pmatrix}_{z=0} \rightarrow \begin{pmatrix} A_{01} \\ B_{01} \end{pmatrix} = T \begin{pmatrix} A_{02} \\ B_{02} \end{pmatrix}$$

$$-\mathrm{Det}(\underline{S}) = r^2 + \tau^2 = 1 \qquad \tau^2 = \tau_{1\to2}\tau_{2\to1}$$

$$T = \frac{1}{\tau_{1\to2}} \begin{pmatrix} 1 & -r \\ -r & rr + \tau^2 \end{pmatrix}$$

A single interface is not very interesting

Example 3.6.2 The Simple Waveguide



$$b_2 = E_o \exp[ik_n(z_o + L)] = a_1 \exp(ik_n L)$$

$$a_2 = E_o \exp[-ik_n(z_o + L)] = b_1 \exp(-ik_n L)$$

$$b_1 = a_2 \exp(ik_n L)$$

scattering matrix

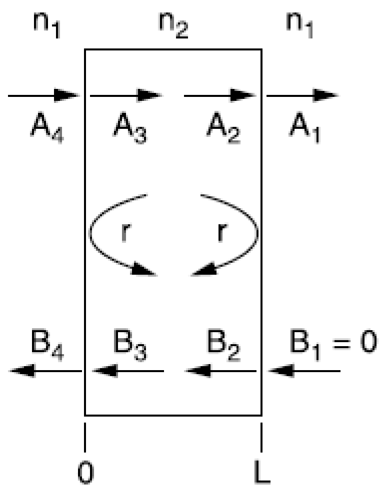
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & \exp(ik_n L) \\ \exp(ik_n L) & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

transfer matrix

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = T \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$$

$$T = \frac{1}{S_{21}} \begin{pmatrix} 1 & -S_{22} \\ S_{11} & -\mathrm{Det}(S) \end{pmatrix} = \frac{1}{\exp(ik_n L)} \begin{pmatrix} 1 & 0 \\ 0 & \exp(2ik_n L) \end{pmatrix} = \begin{pmatrix} \exp(-ik_n L) & 0 \\ 0 & \exp(ik_n L) \end{pmatrix}$$

Example 3.6.3 The Fabry-Perot Cavity



$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \frac{1}{\tau_{21}} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ 0 \end{pmatrix}$$

&

$$\phi = k_n L.$$

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$$

&

$$\begin{pmatrix} A_4 \\ B_4 \end{pmatrix} = \frac{1}{\tau_{12}} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$$

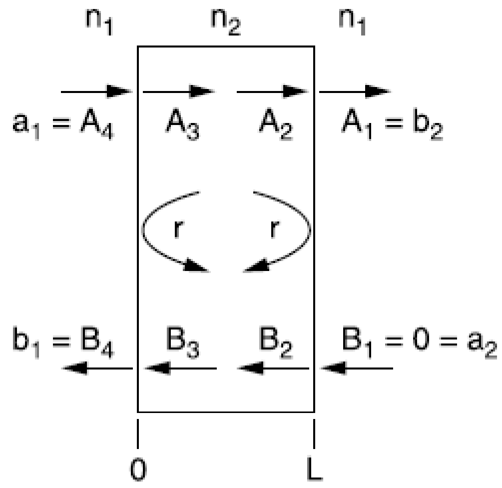
$$\begin{pmatrix} A_4 \\ B_4 \end{pmatrix} = \frac{1}{\tau_{12}} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{+i\phi} \end{pmatrix} \frac{1}{\tau_{21}} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} A_4 \\ B_4 \end{pmatrix} = \frac{1}{\tau^2} \begin{pmatrix} e^{-i\phi} - r^2 e^{+i\phi} & r e^{-i\phi} - r e^{+i\phi} \\ -r e^{-i\phi} + r e^{+i\phi} & -r^2 e^{-i\phi} + e^{+i\phi} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 = 0 \end{pmatrix}$$

How the presence of gain (or loss) affects the TMM

As we know :

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \frac{1}{\tau^2} \begin{pmatrix} e^{-i\phi} - r^2 e^{i\phi} & r\,e^{-i\phi} - r\,e^{i\phi} \\ -r\,e^{-i\phi} + r\,e^{i\phi} & -r^2 e^{-i\phi} + e^{i\phi} \end{pmatrix}$$



$$S = \frac{1}{T_{11}} \begin{pmatrix} T_{21} & \text{Det } T \\ 1 & -T_{12} \end{pmatrix} = \frac{\tau^2}{e^{-i\phi} - r^2 e^{i\phi}} \begin{pmatrix} T_{21} & \text{Det } T \\ 1 & -T_{12} \end{pmatrix}$$

$$b_1 = S_{11} a_1 \quad \Rightarrow \quad \frac{\text{output}}{\text{input}} = \frac{b_1}{a_1} = S_{11} = \frac{-r\,e^{-i\phi} + r\,e^{i\phi}}{e^{-i\phi} - r^2 e^{i\phi}} = -r\,\frac{1 - e^{2i\phi}}{1 - r^2 e^{2i\phi}}$$

از قبل داریم: $$P_o = \frac{g P_{in}}{1 - g/\alpha}$$

بنابراین $$P_{ref} = |S_{11}|^2 P_{in}$$

reflected power

$$|S_{11}|^2 = S_{11}S_{11}^* = r^2 \frac{(1-e^{2i\phi})(1-e^{2i\phi})^*}{(1-r^2 e^{2i\phi})(1-r^2 e^{2i\phi})^*}$$

$$\phi = k_n L$$

Where:

$$k_n = k_o n - i\frac{g_{net}}{2}$$

$$\boxed{g_{net} = \Gamma g - \alpha_{int}}$$

$$\phi = \phi_r + i\phi_i = k_n L = k_o n L - i\frac{g_{net}}{2}L$$

The power reflectivity R is related to the Fresnel reflectivity r (assumed real) $R = r^2$

We define an effective reflectivity R by $\mathscr{R} = \mathscr{R}\exp(-2\phi_i)$

And $(e^{i\phi})^* = e^{-i\phi^*}$

$$|S_{11}|^2 = R\frac{1+\exp(-4\phi_i)-2\exp(-2\phi_i)\cos(2\phi_r)}{1+R^2\exp(-4\phi_i)-2R\exp(-2\phi_i)\cos(2\phi_r)} = R1 + \frac{\frac{\mathscr{R}^2}{R^2} - 2\frac{\mathscr{R}}{R}\cos(2\phi_r)}{1+\mathscr{R}^2 - 2\mathscr{R}\cos(2\phi_r)}$$

$$\cos(2\phi_r) = \cos^2(\phi_r) - \sin^2(\phi_r) = 1 - 2\sin^2(\phi_r)$$

$$P_{\text{ref}} = |S_{11}|^2 P_{\text{in}} = R\frac{\left[1-\frac{R}{R}\right]^2 + 4\,\frac{R}{R}\sin^2\phi_r}{[1-R]^2 + 4R\sin^2\phi_r}P_{\text{in}}$$

Example 3.7.1 A Fabry-Perot Etalon Without Material Gain or Internal Loss

From the previous: $\phi = \phi_r + i\phi_i = k_n L = k_o n L - i\frac{g_{\text{net}}}{2}L = k_o n L$

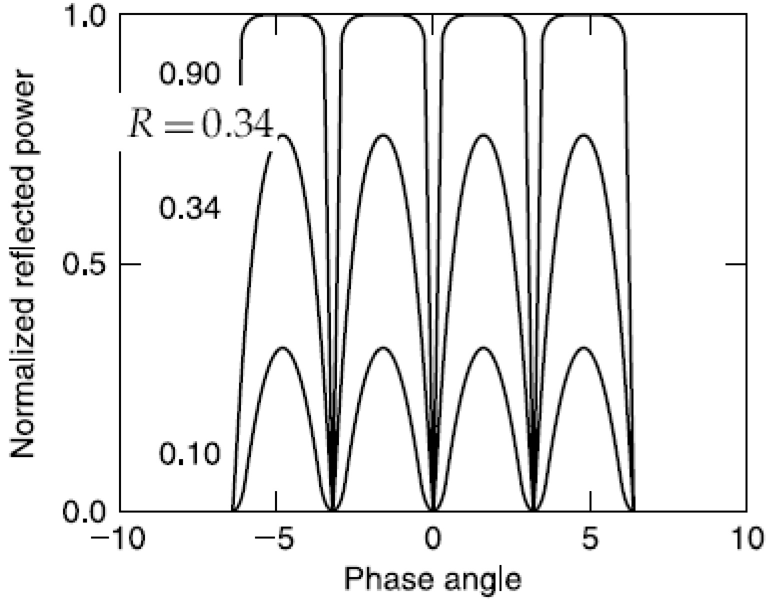
with $R=R$ for real ϕ $\Rightarrow$ $\phi_i = 0$

$$P_{\text{ref}} = |S_{11}|^2 P_{\text{in}} = R\frac{4\,\sin^2(k_o n L)}{[1-R]^2 + 4R\,\sin^2(k_o n L)}P_{\text{in}}$$

$$P_{\text{trans}} = |S_{21}|^2 P_{\text{in}} = \left[1 - |S_{11}|^2\right]P_{\text{in}}$$

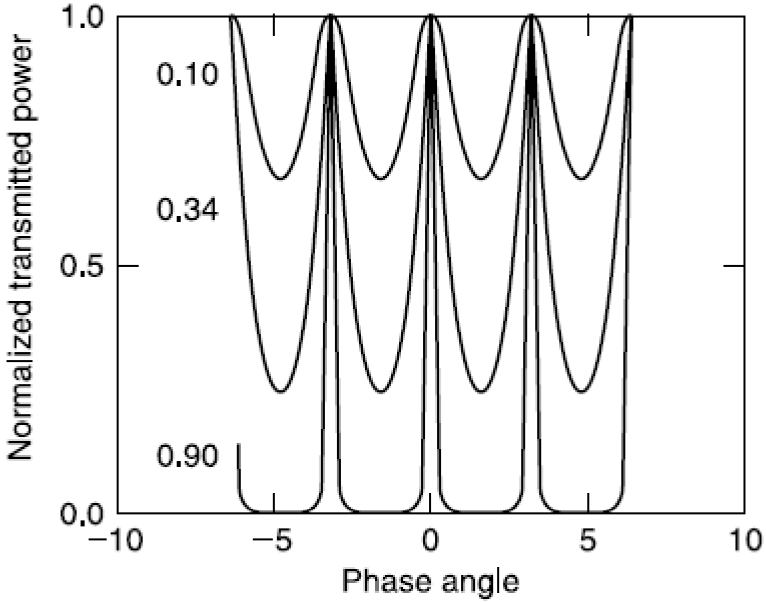
larger facet reflectance R gives larger finesse (narrower line widths)

greater optical loss must lower the finesse of the cavity



phase angle $k_o nL$.

$$k_o = 2\pi / \lambda_o \text{ and } \lambda_o = c/\nu = 2\pi c/\omega$$



Only a very narrow band of wavelengths can propagate through the etalon thereby producing a very narrow bandpass filter

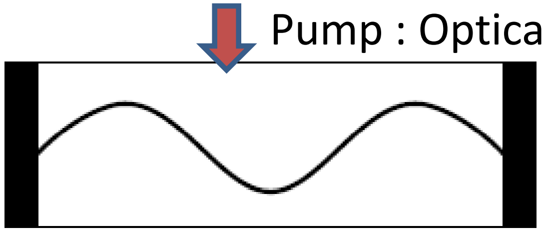
Longitudinal Modes and the Threshold Condition

Pumping the resonator can initiate lasing.

Above threshold, the laser produces considerably **more power than** it receives from the **input beam**.

The effective transmittance and reflectance must become infinite concerning the ring laser and op-amp oscillator. $\bar{S}_{11}$ or $|\bar{S}_{11}|^2$ (etc) equals to zero.

Pump : Optical Power or Current



$$b_1 = S_{11} a_1 \quad \rightarrow \quad P_{\text{ref}} = |S_{11}|^2 P_{\text{in}}$$

$$S_{11} = \frac{-r\,\mathrm{e}^{-i\phi} + r\,\mathrm{e}^{i\phi}}{\mathrm{e}^{-i\phi} - r^2\,\mathrm{e}^{i\phi}} = -r\frac{1 - \mathrm{e}^{2i\phi}}{1 - \mathrm{R}\,\mathrm{e}^{2i\phi}}$$

If: $R = r^2$ $\longrightarrow$ $D = 1 - \mathrm{R}\,\mathrm{e}^{i2\phi}$

$$D = 1 - R\,\exp(g_{\text{net}}L)\,\exp(i2k_o nL) = 1 - R\,\exp(g_{\text{net}}L)\,[\cos(2k_o nL) + i\,\sin(2k_o nL)]$$

D=0 ⇒ **Real part=0**, **Imaginary part=0**

Imaginary Part Produces the Wavelength of the Longitudinal Modes

$$0 = \sin(2k_o nL) = 2\sin(k_o nL)\cos(k_o nL)$$

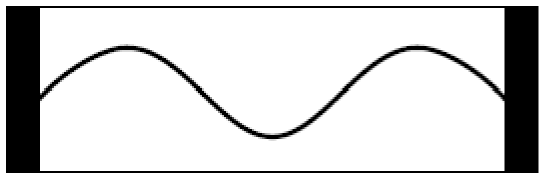
0

$$k_o nL = m\pi \qquad (m = 1, 2, 3, \ldots)$$

$$\lambda_o = \frac{2nL}{m} \qquad \text{or} \qquad \lambda_n = \frac{2L}{m}$$

$$\nu_m = \frac{mc}{2nL} \qquad (m = 1, 2, \ldots)$$

=0 **we would not** necessarily find ***the real part*** of the denominator to be zero



$m = 3$

The m=3 longitudinal mode

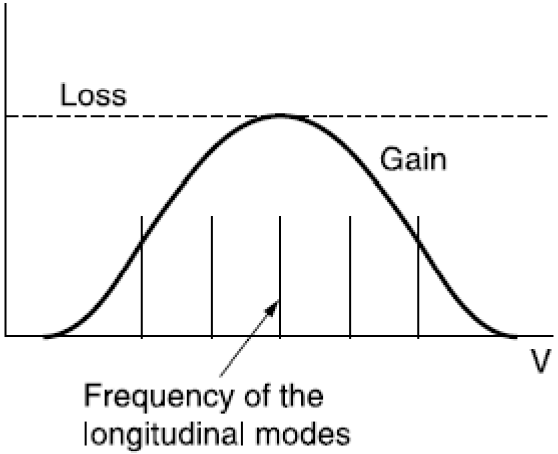
the spacing between adjacent frequencies of the longitudinal modes

$$\nu n = mc/(2L).$$ **Let:** $\Delta m = 1$

$$\Delta(\nu n) = \frac{\Delta mc}{2L} \Rightarrow n\Delta\nu + \nu\Delta n = \frac{c}{2L} \rightarrow \Delta\nu\left[n + \nu\frac{\Delta n}{\Delta\nu}\right] = \frac{c}{2L}$$

Define the group index: $n_g = n + \nu\dfrac{\partial n}{\partial \nu}$

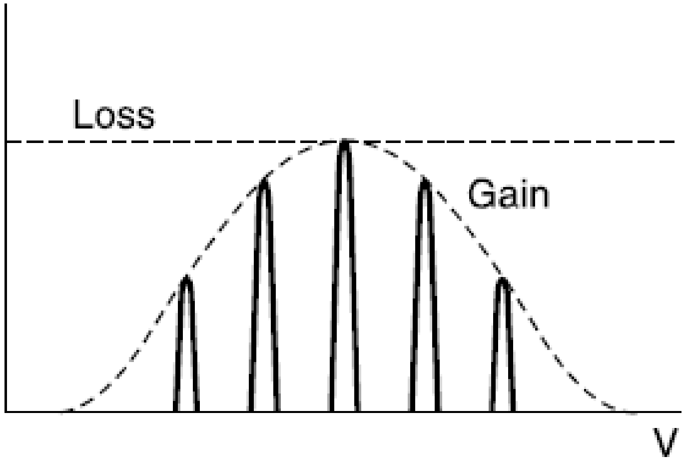
$$\Delta\nu = \frac{c}{2Ln_g} \cong \frac{c}{2Ln_g}$$



If for simplicity we choose n_g =n

$$\nu\lambda_o = c \Rightarrow \lambda_o\Delta\nu + \nu\Delta\lambda_o = 0 \quad \text{or} \quad |\Delta\lambda_o| = \frac{\lambda_o c}{2Ln}$$

The actual line shape pattern appears in:



An extremely important point concerns the role **of homogeneously** vs. **inhomogeneously** broadened lasers.

A **homogeneous broadened laser will have at most one lasing longitudinal mode**.

For an inhomogeneously broadened laser, any number of longitudinal modes can lase.

The reason for these different behaviors has to do with the way the resonator produces the gain

Case 2 The Real Part Produces the Threshold Condition

$$\mathrm{Re}(D) = 1 - R\exp(g_{\mathrm{net}}L)\cos(2k_o nL)$$

1

$$k_o nL = m\pi \qquad (m = 1, 2, 3\ldots)$$

$$0 = 1 - R\exp(g_{\mathrm{net}}L)$$

Where $g_{\mathrm{net}} = \Gamma g - \alpha_{\mathrm{int}}$

operating above threshold

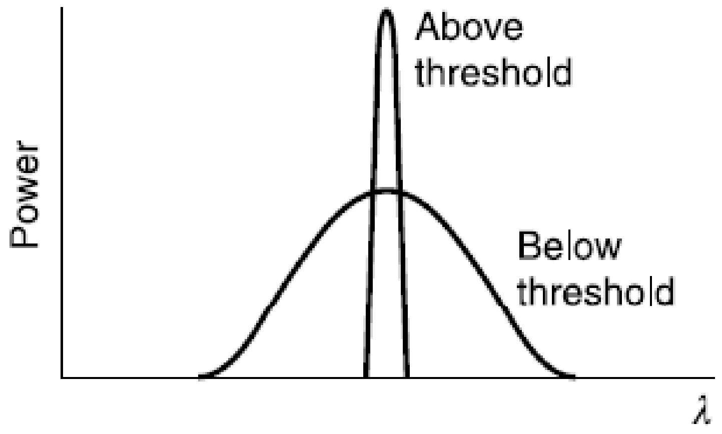
$$g_{\mathrm{net}} = \frac{1}{L}Ln\left(\frac{1}{R}\right) \qquad \text{or} \qquad \Gamma g = \alpha_{\mathrm{int}} + \frac{1}{L}Ln\left(\frac{1}{R}\right)$$

Below threshold this denominator would not be small

Line Narrowing

Define the finesse F in order to discuss the quality of the Fabry-Perot resonator.

$$\mathscr{F} = \frac{4\mathscr{R}_o^2}{(1-\mathscr{R}_o)^2}$$



For convenience, we define an effective reflectivity Ro as

$$\mathscr{R}_o = R\exp(g_{\mathrm{net}}L)$$

large gain should produce large finesse and small line widths

spectrum of emitted power versus wavelength

$$D = 1 - R\,\exp(g_{\text{net}}L)\,\exp(i2k_o nL) = 1 - R\,\exp(g_{\text{net}}L)\cdot[\underbrace{\cos(2k_o nL)}_{1} + i\,\underbrace{\sin(2k_o nL)}_{0}]$$

$$\lambda_n \sim 2L/m \quad\Rightarrow\quad 2k_o nL = 2k_n L = 4\pi L/\lambda_n \sim 2\pi m$$

mode number

$$\sin(2k_o nL) \sim \sin(2\pi m) = 0 \quad \text{and} \quad \cos(2k_o nL) \sim \cos(2\pi m) = 1$$

$$\Downarrow$$

$$D \cong 1 - R\,\exp(g_{\text{net}}L)$$

$$\mathscr{F} = \frac{4\mathscr{R}_o^2}{(1-\mathscr{R}_o)^2}$$

This means the finess is increase while the effective reflective is decrease

Now we show the finesse becomes large and the linewidth decreases as the pumping level increases

steady-state photon laser rate equation

$$0 = \frac{d\gamma}{dt} = v_g \Gamma g \gamma - \gamma v_g(\alpha_{\text{int}} + \alpha_m) + \beta R_{sp}$$

$$g_{\text{net}} = \Gamma g - \alpha_{\text{int}} = \alpha_m + \frac{\beta v_g R_{sp}}{\gamma \uparrow} \downarrow$$

$$1 - \mathscr{R}_o = 1 - R\,\exp(g_{\text{net}}L) = 1 - R\exp(\alpha_m L)\exp\left(\frac{\beta v_g R_{sp}}{\gamma}\right)$$

$$\alpha_m = \frac{1}{L}\,\mathrm{Ln}\left(\frac{1}{R}\right) \qquad R\,\exp(\alpha_m L) = 1$$

$$1 - \mathscr{R}_o = 1 - R\;\exp(g_{\text{net}}L) = 1 - \exp\left(\frac{\beta v_g R_{sp}}{\gamma}\right)$$

$$\mathscr{F} = \frac{4\mathscr{R}^2}{(1-\mathscr{R})^2} \cong \frac{4}{\left[1 - \exp\left(\frac{\beta v_g R_{sp}}{\gamma}\right)\right]^2} \to \infty \quad \text{as} \quad \text{gain} \to \text{loss}$$

as a very important note, **the gain is always infinitesimally smaller than the loss** due to the presence of spontaneous emission.
It should be clear at this point that **spontaneous emission prevents the linewidth from actually becoming zero.**

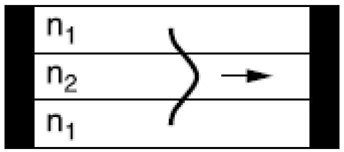
Introduction to Waveguides

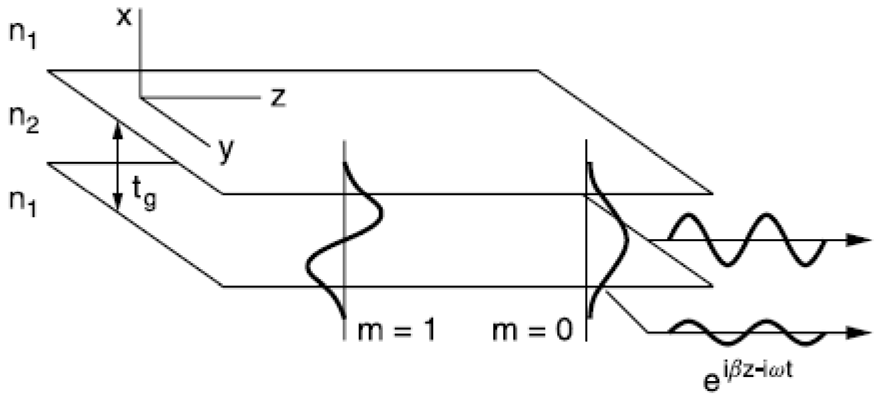
two complimentary approaches

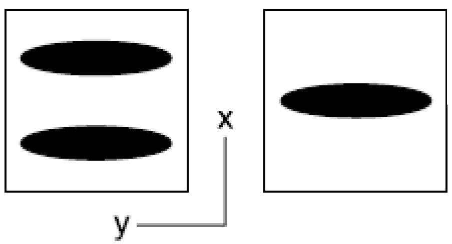
- geometric optics
  - uses rays
- physical optics
  - solves Maxwell's equations

EM Waves for Waveguiding

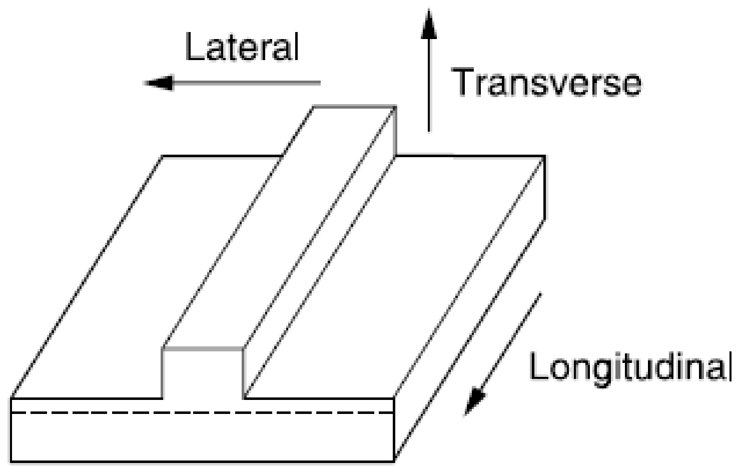
$(n_2 > n_1)$



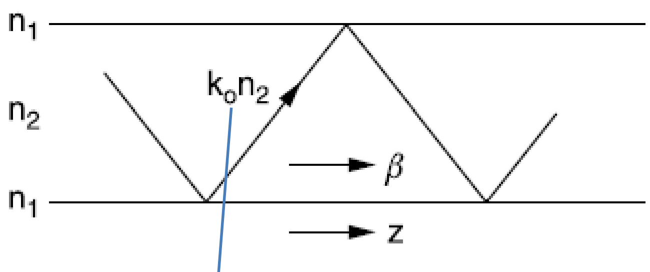




$$e^{i\beta z - i\omega t}$$



The Triangle Relation



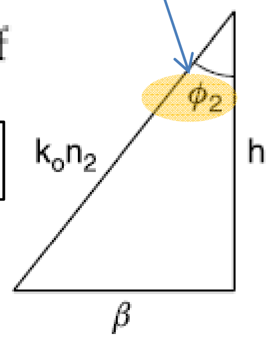
The magnitude of the actual wave vector

; $\beta = k_o n_{eff}$

effective index

$\beta \leq k_o n_2$

only certain values of are allowed

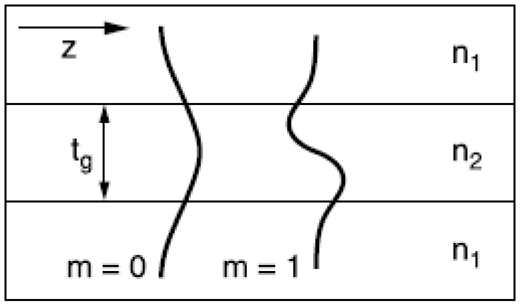


average speed= c/n_{eff}

Only certain values of ''h'' produce standing waves. That is, there are certain allowed wavelengths for the transverse modes approximately given by

$$\lambda_{trans} = 2\pi/h = 2t_g/\,(m+1)$$

As shown, the vector β represents propagation along the length of the waveguide while **the vector ''h'' represents propagation perpendicular to the length of the waveguide**. The quantity ''h'' must also be a wave vector.



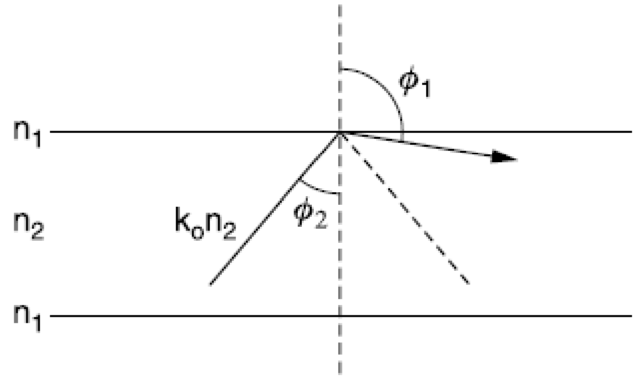
$$\beta^2 + h^2 = (k_o n_2)^2$$

3.8.4 The Cut-off Condition from Geometric Optics

A waveguide can only propagate electromagnetic waves with certain values for the effective wave vector

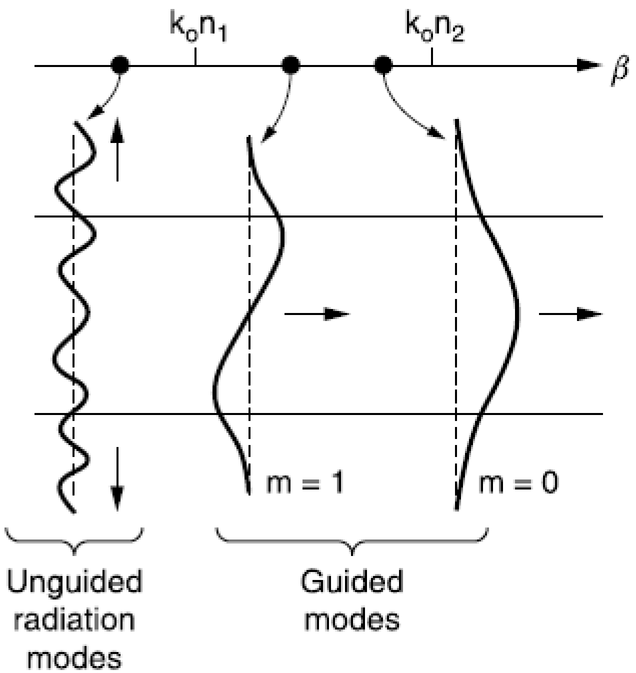
This topic shows how **Snell's law and** the existence of a **critical angle** for **total internal reflection (TIR)** leads to a minimum value for the propagation constant $k_o n_1 \leq \beta$

The smallest value of β is the cutoff value



Combining the cutoff condition and the triangle relation, we find that the effective wave vector must have a magnitude within the range given by

$$\beta_{\min} = k_o n_1 \leq \beta \leq k_o n_2 = \beta_{\max}$$



$$n_1 \leq n_{\text{eff}} \leq n_2.$$

$$\beta = k_o n_{\text{eff}}$$

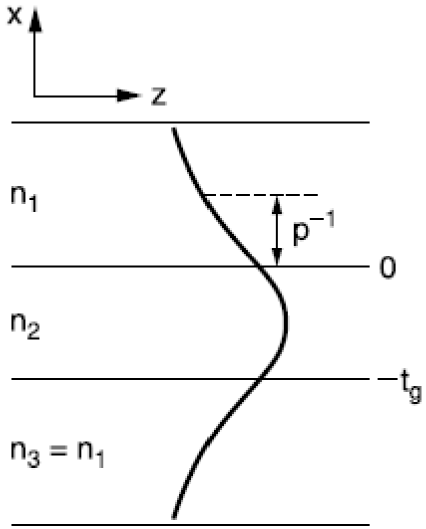
group velocity

$$v_{gw} = \frac{d\omega}{d\beta}$$

$$v_{gw} = \frac{d\omega}{d\beta} = \frac{d\omega}{dk_n}\frac{dk_n}{d\beta} = v_g \frac{dk_n}{d\beta}$$

Physical Optics Approach to Waveguiding

The Wave Equations

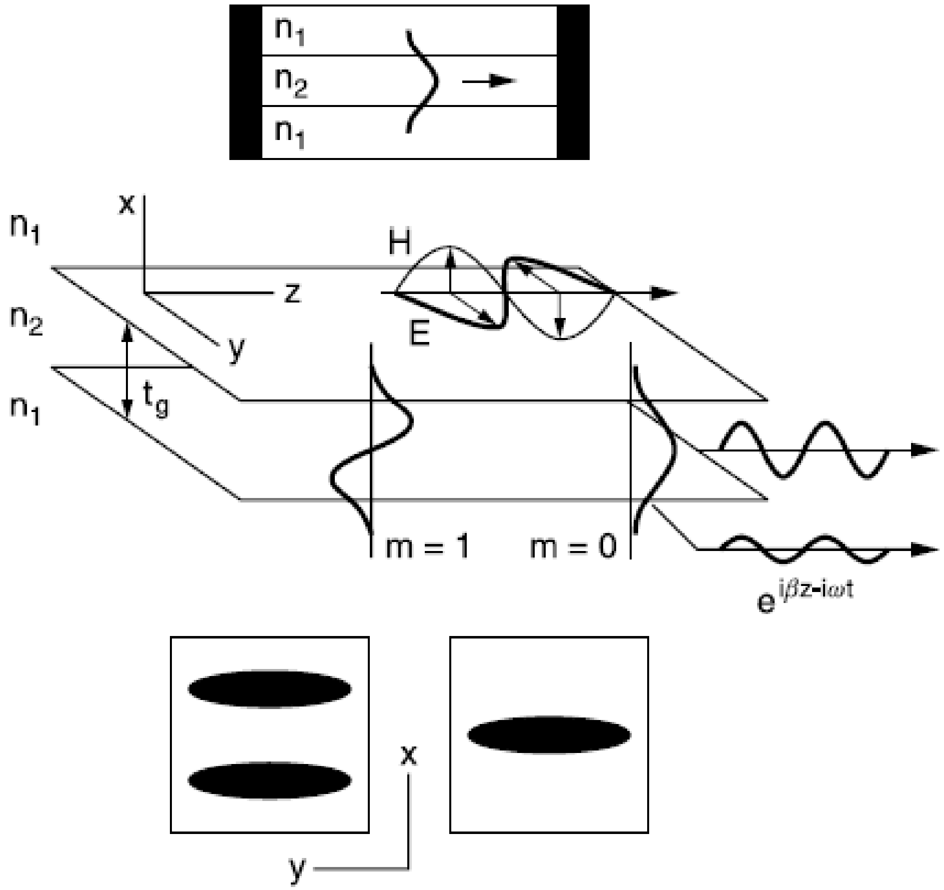


A slab waveguide with the penetration depth $1/p$.

$$\nabla^2 \vec{\mathscr{E}} - \frac{n_j^2}{c^2}\frac{\partial^2 \vec{\mathscr{E}}}{\partial t^2} = 0$$

The General Solutions

$$\vec{\mathscr{E}} = \tilde{y} E_y(x) \exp(i\beta z - i\omega t)$$



$$\frac{\partial^2 E_y}{\partial x^2} + (k_o^2 n_i^2 - \beta^2)E_y = 0$$

$$k_o n_j = \frac{n_j \omega}{c}$$

$$E_y(x) \sim \exp\left(\pm i\, x \sqrt{k_o^2 n_j^2 - \beta^2}\right)$$

Sinusoidal $\quad k_o^2 n_j^2 > \beta^2$

Exponential $\quad k_o^2 n_j^2 < \beta^2$

Region 1 $\quad E_y = A\exp(-px)$

Region 2 $\quad E_y = B\cos(hx) + C\sin(hx)$

Region 3 $\quad E_y = D\exp[p(x + t_g)]$

$$p = \sqrt{\beta^2 - k_o^2 n_1^2} \qquad h = \sqrt{k_o^2 n_2^2 - \beta^2}$$