دانشگاه آزاد
مرودشت

Classical Electromagnetic and Lasers

By: Dr. Jabbari

Maxwell's Equations and Related Quantities

$$\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t} \qquad \nabla \cdot \vec{\mathscr{D}} = \rho_{\text{free}}$$

$$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t} \qquad \nabla \cdot \vec{\mathscr{B}} = 0$$

$\mathscr{J}$ current density

$\mathscr{E}$ **electric field**

$\mathscr{D}$ **displacement field**

$\mathscr{H}$ **magnetic field**

$\mathscr{B}$ **Magnetic induction**

ρ **Charge**

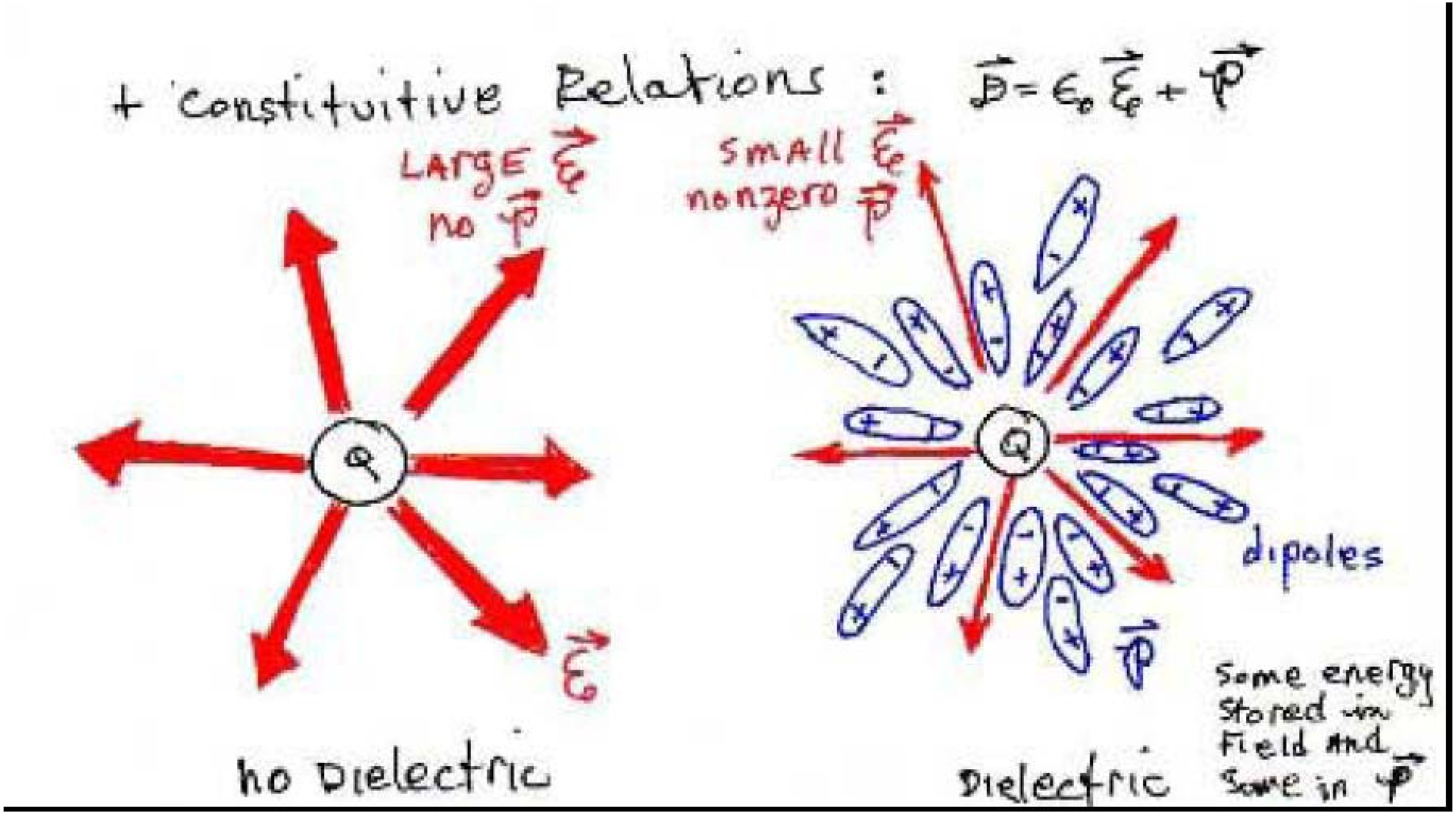
Maxwell's equations

1- find a complex
wave vector kn to describe gain/absorption
and refractive index

2-find the Poynting
vector for electromagnetic power flow

3-develop scattering and linear systems theory
for optical devices

4-develop the theory of optical waveguides

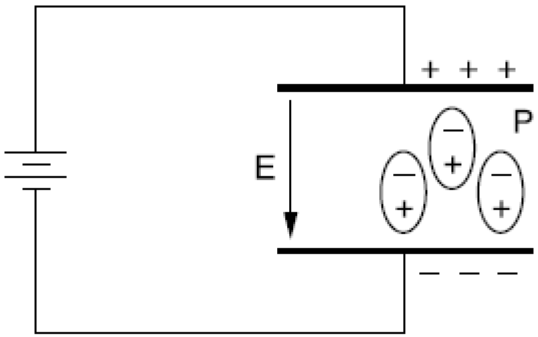

+ Constituitive Relations : $\vec{D} = \epsilon_0 \vec{\mathcal{E}}_e + \vec{P}$
Large $\vec{\mathcal{E}}_e$
no $\vec{P}$
small $\vec{\mathcal{E}}_e$
nonzero $\vec{P}$
Q
$\vec{\mathcal{E}}_e$
no Dielectric
Q
dipoles
$\vec{P}$
Dielectric
Some energy stored in Field and Some in $\vec{P}$

$$\mathscr{E}(\vec{r},t) = E(\vec{r})e^{i\omega t}$$

وابسته به زمان و مکان است که می توان آن را بطور مستقل از زمان تبدیل کرد

$$\vec{r} = x\tilde{x} + y\tilde{y} + z\tilde{z}$$ بردار پایه

$$\vec{\mathscr{D}} = \varepsilon_0\vec{\mathscr{E}} + \vec{\mathscr{P}}$$

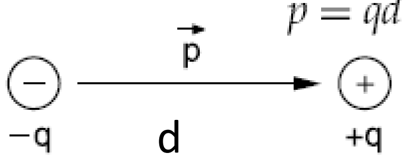


$$\vec{\mathscr{P}}(\vec{r},t)$$

The polarization denotes the **total dipole moment per unit volume** at the position r at time t. The polarization and the dipole moment are related by

$$\vec{P} = \frac{\#\text{dipoles}}{\text{vol}}\,\vec{p}$$

$$\vec{p} = q\vec{d}$$



Linear relation between the induced polarization and the electric field

$$\vec{\mathscr{D}} = \varepsilon_0 \vec{\mathscr{E}} + \vec{\mathscr{P}}$$

$$\vec{\mathscr{P}} = \varepsilon_0 \chi(\omega) \vec{\mathscr{E}}$$

$$\vec{D} = \epsilon \vec{\mathcal{E}}$$

permittivity of free space

(complex) susceptibility

$$\varepsilon = \varepsilon_0(1 + \chi)$$

Gauss' law

$$\nabla \cdot \vec{\mathscr{D}} = \rho_{\text{free}} \rightarrow \varepsilon_0 \nabla \cdot \vec{\mathscr{E}} + \nabla \cdot \mathscr{P} = \rho_{\text{free}}$$

the "**+**" indicates that energy stored as polarization **decreases** the energy stored in the field within a dielectric (since the sum of the two terms equals the **constant free charge**).

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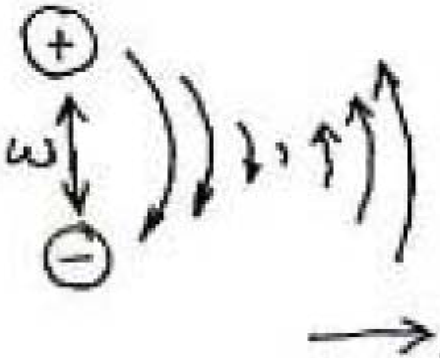
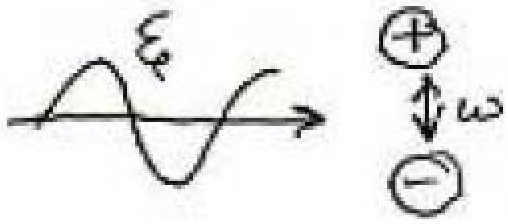
- Dipoles emit and absorb E-M WAVES

Emission



speed = c

Absorption



dissip as damping

- EM WAVE interacts with dipoles
Index of Refraction

relation between the magnetic induction and the magnetic field

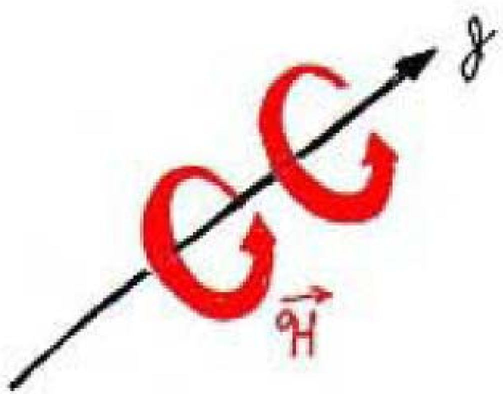
$$\vec{\mathscr{B}} = \mu_0 \vec{\mathscr{H}} + \mu_0 \vec{\mathscr{M}}$$

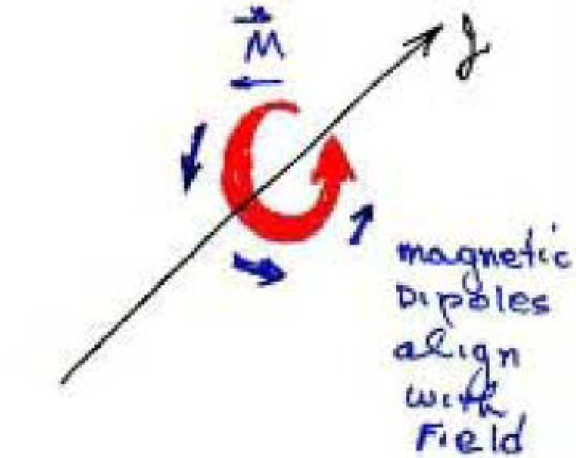
magnetic induction: $\vec{\mathscr{B}}$

permeability of free space: μ_0

magnetic fields: $\vec{\mathscr{H}}$

material magnetization: $\vec{\mathscr{M}}$





معمولا از M صرف نظر میگردد

non magnetic

Large $\vec{H}$

Small M

$\vec{B} = \mu_0 \vec{H}$

magnetic

Large $\vec{H}$

Large $\vec{M}$

Large $\vec{B} > \mu_0 \vec{H}$

=0 for steady state conditions

$$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t} \rightarrow \nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} \rightarrow \boxed{\nabla \times \vec{\mathscr{B}} - \mu_o \nabla \times \vec{\mathscr{M}} = \mu_o \vec{\mathscr{J}}}$$

current increases, both the magnetization and the magnetic induction also increase

Outside the magnetic material

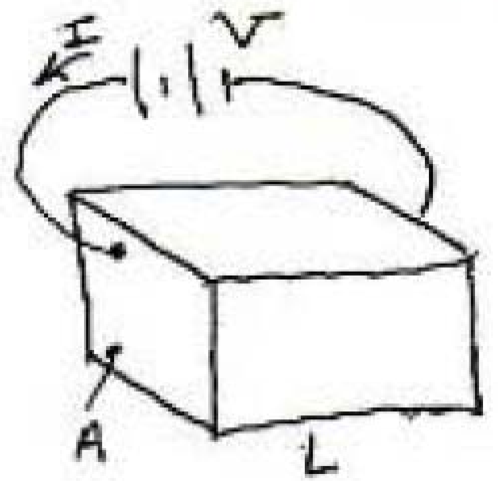
$$\vec{\mathscr{B}} = \mu_o \vec{\mathscr{H}}$$

conductivity of the material

Ohm's law.

$$\vec{\mathscr{J}} = \sigma \vec{\mathscr{E}}$$

Ohm's law.



σ = Conductivity

ρ = Resistivity

$$\sigma = \frac{1}{\rho}$$

$$JA = \sigma \mathcal{E} A$$

$$= \sigma A \frac{\mathcal{E} L}{L}$$

$$J \sim \frac{\text{current}}{\text{area}}$$

$$I = \frac{\sigma A}{L} V$$

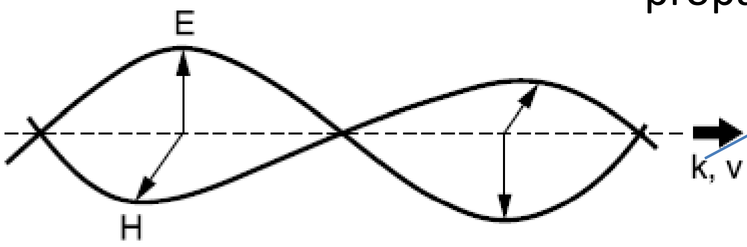
$$R = \frac{L}{\sigma A} = \rho \frac{L}{A}$$

Relation between Electric and Magnetic Fields in Vacuum

The vector $\vec{\mathscr{E}} \times \vec{\mathscr{H}}$

propagation direction of the wave

wave vector



We start by plane-wave:

$$\vec{\mathscr{E}}(z,t) = E_o\, e^{ik_o z - i\omega t}\, \tilde{x}$$

$$\vec{\mathscr{H}}(z,t) = H_o\, e^{ik_o z - i\omega t}\, \tilde{y}$$

با استفاده از یکی از معادلات ماکسول شروع می کنیم

$$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t}$$

$$\vec{\mathscr{D}} = \varepsilon_0 \vec{\mathscr{E}} + \vec{\mathscr{P}}$$

برای یک فضای آزاد داریم: $\vec{\mathscr{P}} = 0$

$$\nabla \times \vec{\mathscr{H}} = \varepsilon_0 \frac{\partial \vec{\mathscr{E}}}{\partial t}$$

$$\nabla \times \vec{\mathscr{H}} = \begin{vmatrix} \tilde{x} & \tilde{y} & \tilde{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathscr{H}_x & \mathscr{H}_y & \mathscr{H}_z \end{vmatrix} = \tilde{x}\left(\frac{\partial \mathscr{H}_z}{\partial y} - \frac{\partial \mathscr{H}_y}{\partial z}\right) - \tilde{y}\left(\frac{\partial \mathscr{H}_z}{\partial x} - \frac{\partial \mathscr{H}_x}{\partial z}\right) + \tilde{z}\left(\frac{\partial \mathscr{H}_y}{\partial x} - \frac{\partial \mathscr{H}_x}{\partial y}\right)$$

Only for y-component, we can simplify the relation:

$$\nabla \times \vec{\mathscr{H}} = -\tilde{x}\frac{\partial \mathscr{H}_y}{\partial z} = -i\tilde{x}k_o H_o e^{+ik_o z - i\omega t}$$

The time derivative

$$\frac{\partial}{\partial t}\varepsilon_o \vec{\mathscr{E}}(z,t) = -i\omega\varepsilon_o \tilde{x}E_o\, e^{ik_o z - i\omega t}$$

Substituting

$$k_o H_o = \omega\varepsilon_o E_o \quad \rightarrow \quad H_o = \frac{\omega\varepsilon_o}{k_o}E_o$$

$$B_o = \mu_o H_o$$

$$c^2 = (\varepsilon_o\mu_o)^{-1}$$

$$B_o = \frac{E_o}{c}$$

Relation between Electric and Magnetic Fields in Dielectrics

By the same way:

$$\vec{\mathscr{E}} = E_1\, e^{ikz-i\omega t}\,\tilde{x} \qquad \vec{\mathscr{H}} = H_1\, e^{ikz-i\omega t}\,\tilde{y}$$

$$k = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\lambda_o/n} = k_o n$$

λ_n: in the medium

λ_o: in vacuum

n: real index of refraction

$$\nabla \times \vec{\mathscr{H}} = \frac{\partial}{\partial t}\vec{\mathscr{D}} = \frac{\partial}{\partial t}\left(\varepsilon_o\vec{\mathscr{E}} + \vec{\mathscr{P}}\right) = \frac{\partial}{\partial t}\left(\varepsilon_o\vec{\mathscr{E}} + \varepsilon_o\chi\vec{\mathscr{E}}\right)$$

$$\varepsilon = \varepsilon_o(1+\chi)$$

$$\nabla \times \vec{\mathscr{H}} = \frac{\partial}{\partial t}\left(\varepsilon\vec{\mathscr{E}}\right)$$

The cross product and derivative can be performed in the same manner as above to obtain

$$-i\tilde{x}kH_1\, e^{ikz-i\omega t} = -i\tilde{x}\omega\varepsilon E_1\, e^{ikz-i\omega t}$$

$$H_1 = \frac{\varepsilon\omega}{k_o n}E_1$$

neglects the magnetization

$$M = 0 \Rightarrow B_1 = \frac{\omega\mu_o\varepsilon}{k_o n}E_1$$

$$v = \frac{1}{\sqrt{\mu_o\varepsilon}} = \frac{c}{n} \quad \rightarrow \quad \mu_o\varepsilon = \frac{n^2}{c^2}$$

$$c = \frac{\omega}{k_o}$$

$$\Rightarrow B_1 = \frac{c\mu_o\varepsilon}{n}E_1 = \frac{c(n/c)^2}{n}E_1 = \frac{n}{c}E_1 = \frac{E_1}{v}$$

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The Wave Equation

Derivation of the Wave Equation

$$\left\{\begin{aligned} &\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t} \\ &\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t} \end{aligned}\right.$$

$\nabla \times$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = -\frac{\partial}{\partial t} \nabla \times \vec{\mathscr{B}} = -\frac{\partial}{\partial t} \nabla \times (\mu_o \vec{\mathscr{H}})$$

$\vec{\mathscr{J}} = \sigma \vec{\mathscr{E}}$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = -\mu_o \frac{\partial}{\partial t}\left(\vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t}\right) = -\mu_o \frac{\partial \vec{\mathscr{J}}}{\partial t} - \mu_o \frac{\partial^2 \vec{\mathscr{D}}}{\partial t^2} = -\mu_o \sigma \frac{\partial \vec{\mathscr{E}}}{\partial t} - \mu_o \frac{\partial^2}{\partial t^2}\left(\varepsilon_o \vec{\mathscr{E}} + \varepsilon_o \chi \vec{\mathscr{E}}\right)$$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = -\mu_o \sigma \frac{\partial \vec{\mathscr{E}}}{\partial t} - \mu_o \frac{\partial^2}{\partial t^2}\left(\varepsilon \vec{\mathscr{E}}\right)$$

$$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = \nabla(\nabla \cdot \vec{\mathscr{E}}) - \nabla^2 \vec{\mathscr{E}}$$

$$\nabla^2 \vec{\mathscr{E}} = \mu_o \sigma \frac{\partial \vec{\mathscr{E}}}{\partial t} + \mu_o \varepsilon_o (1 + \chi) \frac{\partial^2 \vec{\mathscr{E}}}{\partial t^2}$$

The Complex Wave Vector

We start by substituting a plane wave into the wave equation

electric field consists of a single traveling plane wave

$$\vec{\mathscr{E}} = \tilde{e}E_o \exp(ik_n z - i\omega t)$$

$$k_c = k_r + i\,k_i = \mathrm{Re}(k_c) + i\,\mathrm{Im}(k_c)$$

اگر همان مسیری را که برای حالت قبل رفته بودیم در اینجا با بردار موج مختلط
انجام دهیم نتیجه زیر بدست خواهد آمد

$$-k_c^2 + i\mu_o\sigma\omega + \mu_o\varepsilon_o\omega^2(1+\chi) = 0$$

$$c = \frac{1}{\sqrt{\varepsilon_o\mu_o}} \qquad c = \frac{\omega}{k_o} \rightarrow k_o = \frac{\omega}{c}$$

$$k_c^2 = \frac{\omega^2}{c^2}(1+\chi) + i\mu_o\sigma\omega = k_o^2(1+\chi) + i\mu_o\sigma\omega$$

$$k_c^2 = k_o^2(1+\chi) + i\frac{k_o^2}{k_o^2}\mu_o\sigma\omega = k_o^2(1+\chi) + ik_o^2\frac{\mu_o\sigma\omega}{(\omega/c)^2} = k_o^2(1+\chi) + ik_o^2\frac{\sigma}{\varepsilon_0\omega}$$

$$k_c^2 = k_o^2\left[1+\chi+i\frac{\sigma}{\varepsilon_0\omega}\right]$$

$$k_c = k_o\left[1+\mathrm{Re}(\chi)+i\left(\mathrm{Im}(\chi)+\frac{\sigma}{\varepsilon_0\omega}\right)\right]^{1/2}$$

Definitions for Complex Index, Permittivity and Wave Vector

complex refractive index and the complex permittivity

$$n_c = n_r + i\,n_i$$

$$\varepsilon_c = \varepsilon_r + i\,\varepsilon_i$$

$$k_c = k_o n_c$$

$$n_c^2 = 1 + \mathrm{Re}(\chi) + i\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right)$$

معمولا قسمت حقیقی بدون اندیس نوشته می شود

$$n^2 = \varepsilon/\varepsilon_o \quad \text{or} \quad n = \sqrt{\varepsilon/\varepsilon_o}$$

$$\frac{\varepsilon_r}{\varepsilon_o} + i\frac{\varepsilon_i}{\varepsilon_o} = 1 + \mathrm{Re}(\chi) + i\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right)$$

$$\frac{\varepsilon_r}{\varepsilon_o} = 1 + \mathrm{Re}(\chi) \qquad \frac{\varepsilon_i}{\varepsilon_o} = \mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}$$

So real permittivity and real refractive index are related to the real part of the susceptibility.

imaginary of the permittivity is related to both the imaginary part of the susceptibility and the conductivity

Conductivity → absorbs part of the electromagnetic wave

another definition

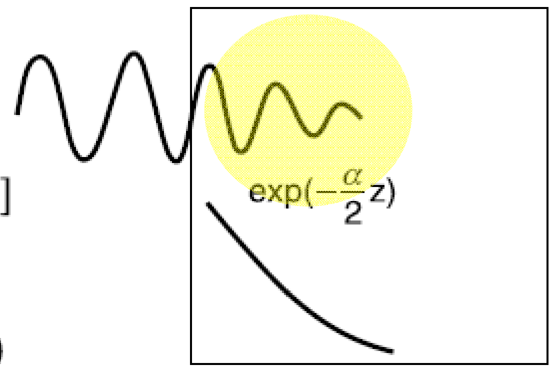
$$k_c = k_o n_c = k_o n + i\frac{\alpha}{2} = k_o n - i\frac{g_n}{2}$$

absorption and the gain

The Meaning of k_n

the imaginary part of the wave vector gives the exponential decay or growth (absorption or gain, respectively) of the traveling wave

$$\mathrm{Re}(k_c) = k_o n_r > k_o \quad \rightarrow \quad \lambda_{\text{medium}} < \lambda_{\text{vacuum}}$$



$$E = E_o \exp(ik_c z) = E_o \exp[i(k_o n + i\alpha/2)z)] = E_o \exp(-z\alpha/2)\exp[ik_o nz]$$

The power has the form of $P \sim E^* E \sim \exp(-\alpha z) = \exp(+gz)$

Approximate Expression for the Wave Vector

For simplicity

$$k_c = k_o n_c = k_o\left[1 + \mathrm{Re}(\chi) + i\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_0 \omega}\right)\right]^{1/2} \xrightarrow{n_r^2 = 1 + \mathrm{Re}(\chi)} k_c = k_o n_c = k_o n_r\left[1 + \frac{i}{n_r^2}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_0 \omega}\right)\right]^{1/2}$$

$$\xrightarrow{\sqrt{1+y} \approx 1 - \frac{y}{2}} \quad k_c = k_o n_c = k_o n_r + i\frac{k_o}{2n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_0 \omega}\right)$$

$$k_c = k_o n + i\alpha/2$$

very important result

$$\alpha = \frac{k_o}{n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_0 \omega}\right) = -g_n$$

$$\alpha = -g_n$$

Approximate Expressions for the Refractive Index and Permittivity

$$n_c = \sqrt{\frac{\varepsilon_c}{\varepsilon_o}} = \sqrt{\frac{\varepsilon_r}{\varepsilon_o} + i\frac{\varepsilon_i}{\varepsilon_o}}$$

imaginary part of **ε** is small

$$\tilde{n} = \sqrt{\frac{\varepsilon_r}{\varepsilon_o}}\left[1 + i\frac{\varepsilon_i/\varepsilon_o}{\varepsilon_r/\varepsilon_o}\right]^{1/2} \cong n_r\left[1 + i\frac{\varepsilon_i/\varepsilon_o}{2\varepsilon_r/\varepsilon_o}\right] = n_r\left[1 + i\frac{\varepsilon_i/\varepsilon_o}{2n_r^2}\right] = n_r + i\frac{\varepsilon_i/\varepsilon_o}{2n_r}$$

$$\tilde{n} = n_r + i\, n_i$$

$$n_r = \sqrt{\frac{\varepsilon_r}{\varepsilon_o}} \qquad n_i = \frac{\varepsilon_i/\varepsilon_o}{2n_r}$$

$$n_r = \sqrt{1 + \mathrm{Re}(\chi)} \qquad n_i = \frac{1}{2n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o\omega}\right)$$

$$\varepsilon_i = \varepsilon_o\ \mathrm{Im}(\chi) + \frac{\sigma}{\omega}$$

The Susceptibility and the Pump

The **real part of** the susceptibility leads to the **index of refraction** while the **imaginary part** leads to **absorption or gain** as can be seen from the main two results

$$n = n_r = [1 + \mathrm{Re}(\chi)]^{1/2}$$

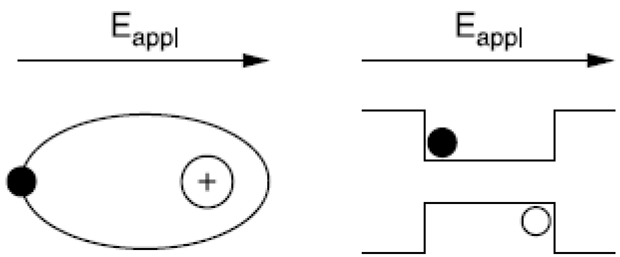
$$\alpha = \frac{k_o}{n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right) = -g_n$$

The **pump** mechanism adds energy to **the laser**

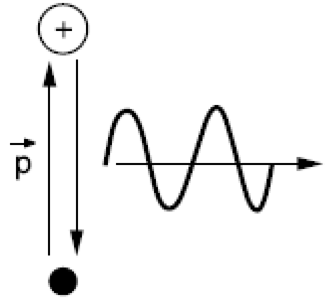
It is the **susceptibility t**hat changes with **pumping** very similar to p**olarization** $P = \varepsilon_o \chi E$

Adding **carriers through** the pump mechanism increases the number of **possible dipoles**

$$\chi = \chi_b + \chi_p.$$



$$\alpha = \frac{k_o}{n_r}\left(\mathrm{Im}(\chi_b) + \mathrm{Im}(\chi_p) + \frac{\sigma}{\varepsilon_o \omega}\right) = \frac{k_o}{n_r}\frac{\sigma}{\varepsilon_o \omega} + \frac{k_o}{n_r}\mathrm{Im}(\chi_b + \chi_p) = \alpha'_{\mathrm{int}} - g$$



related to the term containing the conductivity

related to the term containing the susceptibility

Next

$$n = n_r = \left[1 + \mathrm{Re}(\chi_b) + \mathrm{Re}(\chi_p)\right]^{1/2}$$

Define the background refractive index $n_b = \sqrt{1 + \mathrm{Re}(\chi_b)}$.

مهّم

$$n = n_b\left[1 + \frac{\mathrm{Re}(\chi_p)}{n_b^2}\right]^{1/2} \cong n_b\left[1 + \frac{\mathrm{Re}(\chi_p)}{2n_b^2}\right] = n_b + \frac{\mathrm{Re}(\chi_p)}{2n_b}$$

We see that the refractive **index is smaller than the background** refractive index **when** the real part of the pump susceptibility **is negative**.

the pump susceptibility changes the **index of refraction**