References

LIGHTWAVE TECHNOLOGY
Telecommunication Systems

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Physics of OPTOELECTRONICS

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SHUN LIEN CHUANG

PHYSICS OF OPTOELECTRONIC DEVICES

PAUL HARRISON

Quantum Wells, Wires and Dots

2nd Edition

LIGHTWAVE TECHNOLOGY
Telecommunication Systems

4 Nonlinear Impairments

5 Signal Recovery and Noise

6 Optical Amplifier Noise

7 Dispersion Management

8 Nonlinearity Management

10 Optical Networks

Chapter 2, 3 Basic concept
Chapter 4 Mathematical Foundations

Chapter 5 Fundamentals of Dynamics

Chapter 6 Light

Chapter 7 Matter–Light Interaction

Chapter 8 Semiconductor Emitters and Detectors

Physics of
OPTOELECTRONICS
Michael A. Parker

دانشگاه آزاد اسلامی

Chuang

Ch 3, Basic quantum mechanics

Ch 4, Theory of electronic band structures in semiconductors

Ch 6, Light propagation in anisotropic media and radiation

Ch 11, Advanced semiconductor lasers

Ch 12, Electrooptic and acoustooptic modulators

Ch 13, Electroabsorption modulators

Ch 15, Avalanche PD and intersubband QW PD

Peyghambarian

Ch 9, Quasi-One- and Zero-Dimensional Semiconductors: Quantum Wires and Quantum Dots

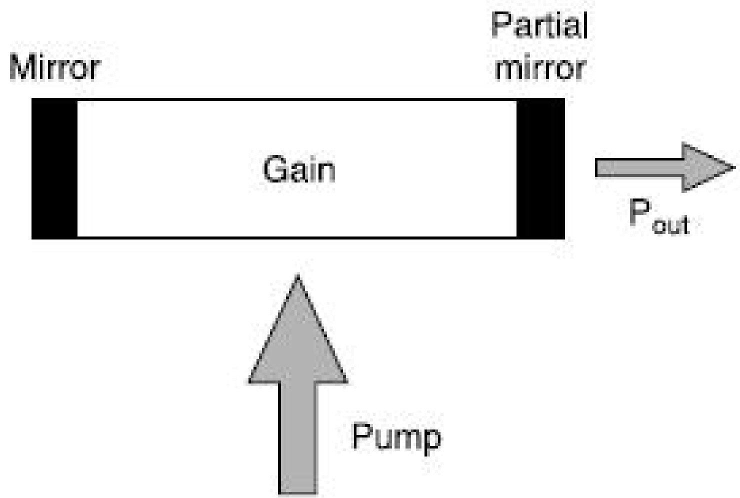
Ch 10, Electro-Optical Properties of Semiconductors

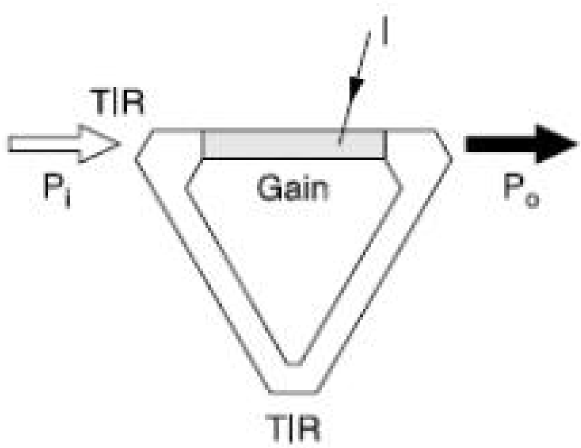
Ch 13, Semiconductor Optical Nonlinearities

Ch 18, Optoelectronic Devices

Introduction
Review of Optoelectronic 2

Basic Components and the Role of Feedback





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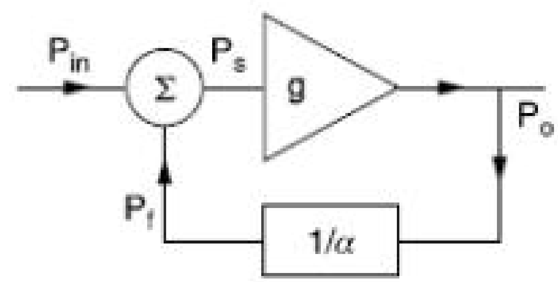
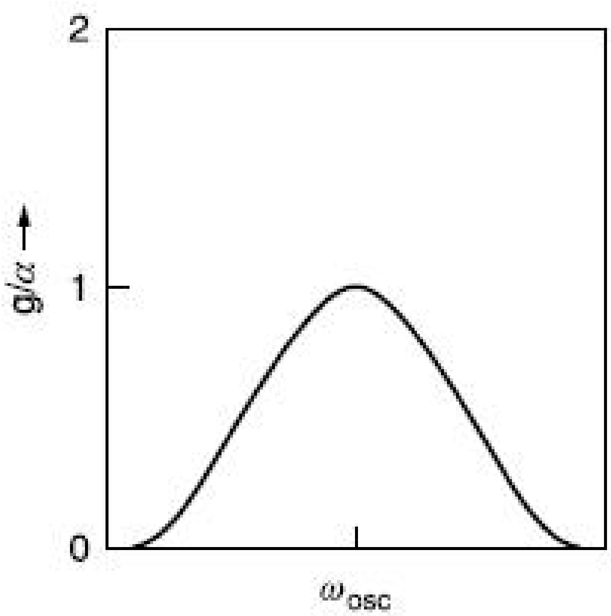


$$\left\{ \begin{array}{c} P_o = gP_s \\ P_s = P_{in} + P_f \\ P_f = \dfrac{1}{\alpha} P_o \end{array} \right\} \quad \rightarrow \quad P_o = \frac{gP_{in}}{1-(g/\alpha)}$$

Unlike Op-Amp laser has g=g(constant)

+ FREQ Dependent Gain

$$G = G(\omega)$$



oscillates at Frequency ω_0 Since $g(\omega_0) \approx \beta$

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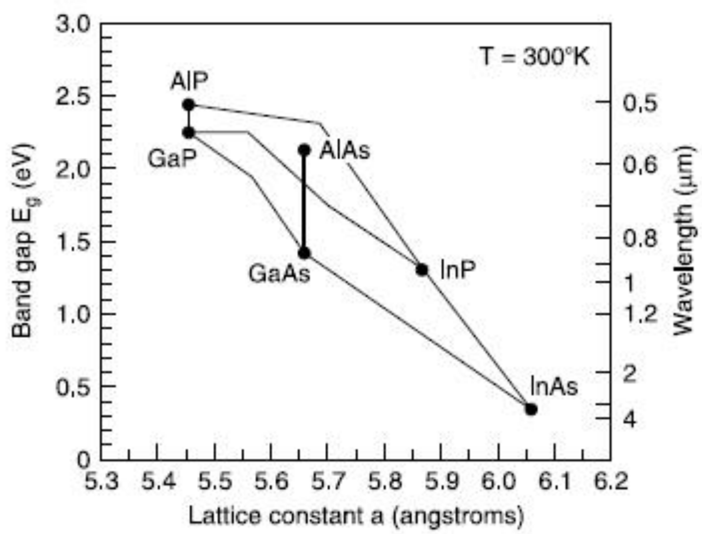
Basic Properties of Lasers

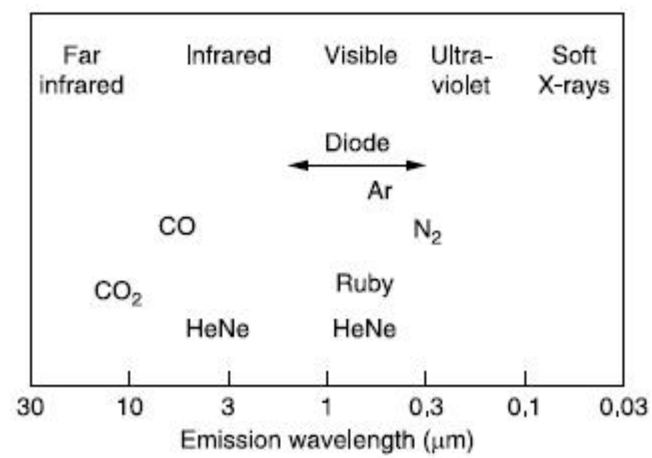
The "brightness," defined as the amount of output power per unit frequency, determines the spectral purity of the source.

Wavelength and Energy

$$E = \hbar\omega = h\nu \quad \lambda = 2\pi/k \quad c = \nu\lambda = \omega/k$$

$$E \doteq 1240/\lambda.$$



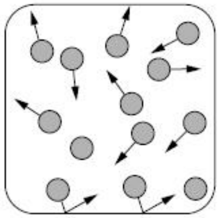


By: Dr. Jabbari

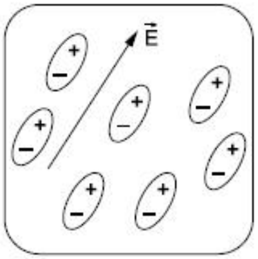
Introduction to Matter and Bonds

Classification of Matter

- Gases

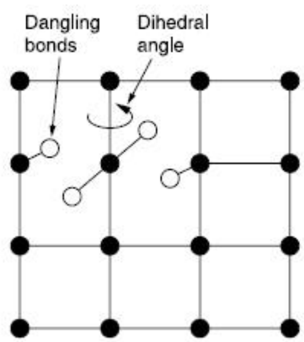


- Liquids and Liquid Crystals



- **Solids**

Bonding and the Periodic Table



Periods	I A	II A	III B	IV B	V B	VI B	VII B	VIII	VIII	VIII	I B	II B	III A	IV A	V A	VI A	VII A	0
1	1,0079 H[1]																1,0079 H[1]	4,00260 He[2]
2	6,941 Li[3]	9,01218 Be[4]											10,81 B[5]	12,011 C[6]	14,0067 N[7]	15,9994 O[8]	18,9984 F[9]	20,179 Ne[10]
3	22,9898 Na[11]	24,305 Mg[12]											26,9815 Al[13]	23,086 Si[14]	30,9738 P[15]	32,06 S[16]	35,453 Cl[17]	39,948 Ar[18]
4	39,098 K[19]	40,08 Ca[20]	44,9559 Sc[21]	47,90 Ti[22]	50,9414 V[23]	51,996 Cr[24]	54,9380 Mn[25]	55,847 Fe[26]	58,9332 Co[27]	58,71 Ni[28]	63,546 Cu[29]	65,38 Zn[30]	69,72 Ca[31]	72,59 Ge[32]	74,9216 As[33]	78,96 Se[34]	79,904 Br[35]	83,80 Kr[36]
5	85,4678 Rb[37]	87,62 Sr[38]	88,9059 Y[39]	91,22 Zr[40]	92,9064 Nb[41]	95,94 Mo[42]	98,9062 Tc[43]	101,07 Ru[44]	102,9055 Rh[45]	106,4 Pd[46]	107,868 Ag[47]	112,40 Cd[48]	114,82 In[49]	118,69 Sn[50]	121,75 Sb[51]	127,60 Te[52]	126,9045 I[53]	131,30 Xe[54]
6	132.9054 Cs[55]	137.34 Ba[56]	[57-71]	178.49 Hf[72]	180.9479 Ta[73]	183.85 W[74]	186.2 Re[75]	190.2 Os[76]	192.22 Ir[77]	195.09 Pt[78]	196.9665 Au[79]	200.59 Hg[80]	204.37 Tl[81]	207.2 Pb[82]	208.9804 Bi[83]	(210) Po[84]	(210) At[85]	(222) Rn[86]
7	(223) Fr[87]	226.0254 Ra[88]	[89-103]	[104]	[105]	[106]	[107]	[109]										

By: Dr. Jabbari

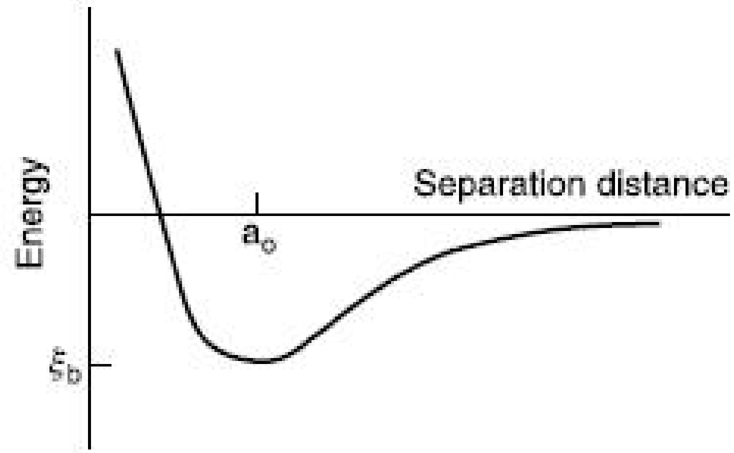
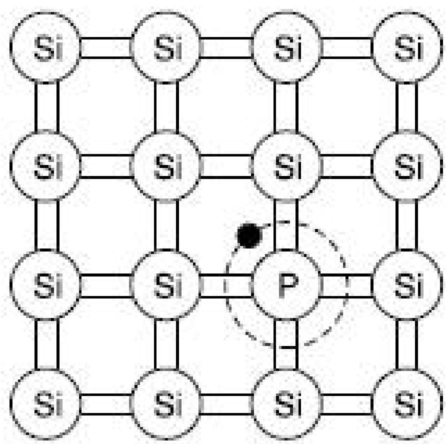


FIGURE 1.4.7
Total energy of two atoms as a function of their separation distance.



An *n*-type dopant atom embedded in a silicon host crystal.

Introduction to Bands and Transitions

Intuitive Origin of Bands

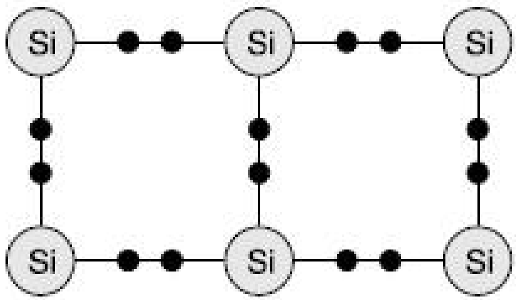


FIGURE 1.5.1
Cartoon representation of silicon crystal at 0 K.

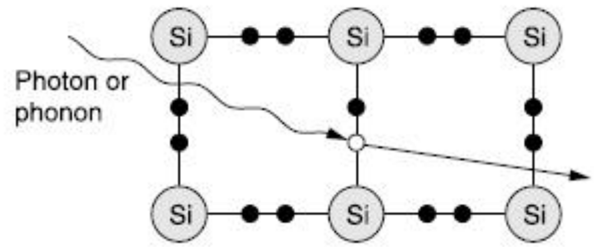


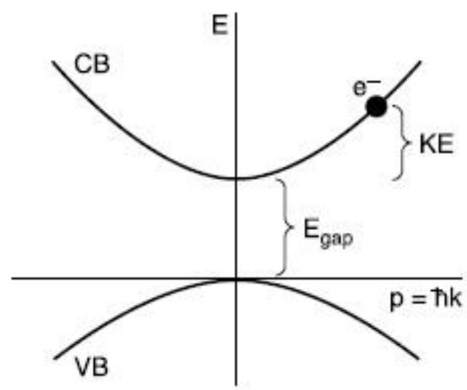
FIGURE 1.5.2
Cartoon representation of transition from VB to CB.

The total energy of a conduction electron can be written as

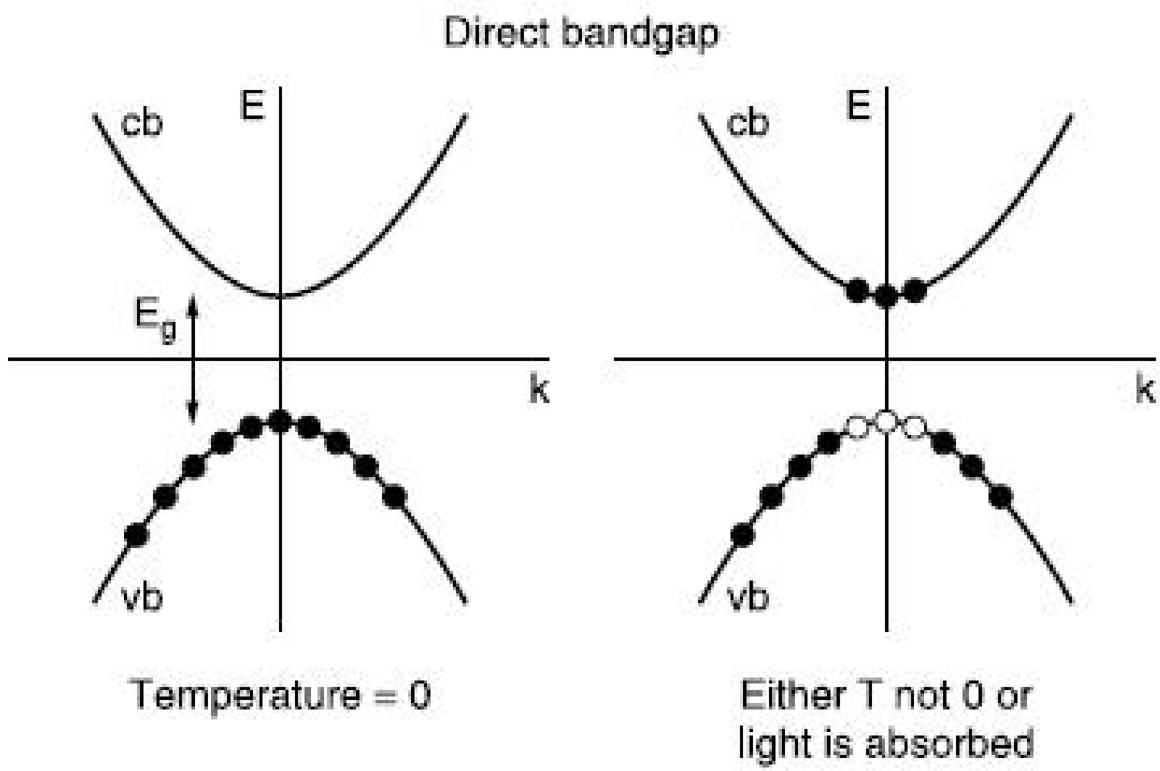
$$E = \mathrm{PE} + \mathrm{KE} = E_g + \tfrac{1}{2} m_e v^2$$

$$p = m_e v$$

$$E = E_g + \frac{p^2}{2m_e}$$



$$E = \frac{p^2}{2m_h}$$



An elementary band diagram for gallium arsenide

Indirect Bands, Light and Heavy Hole Bands

$$p = \hbar k = m_e v.$$

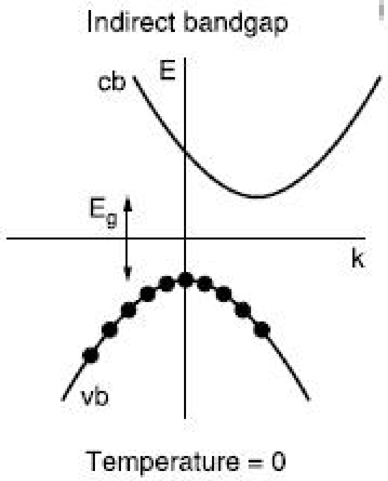


FIGURE 1.5.5
A semiconductor at zero degrees Kelvin with an indirect bandgap.

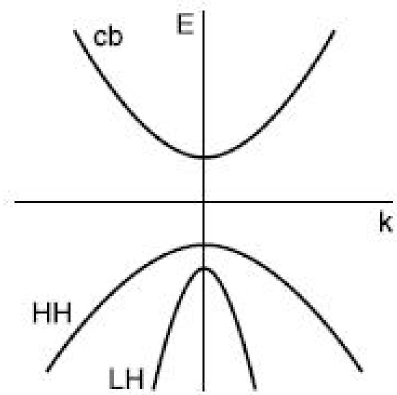
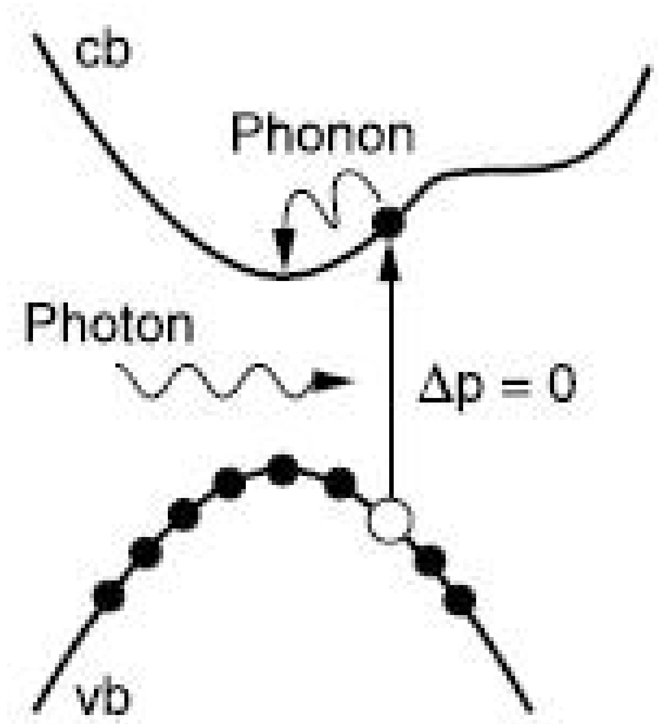
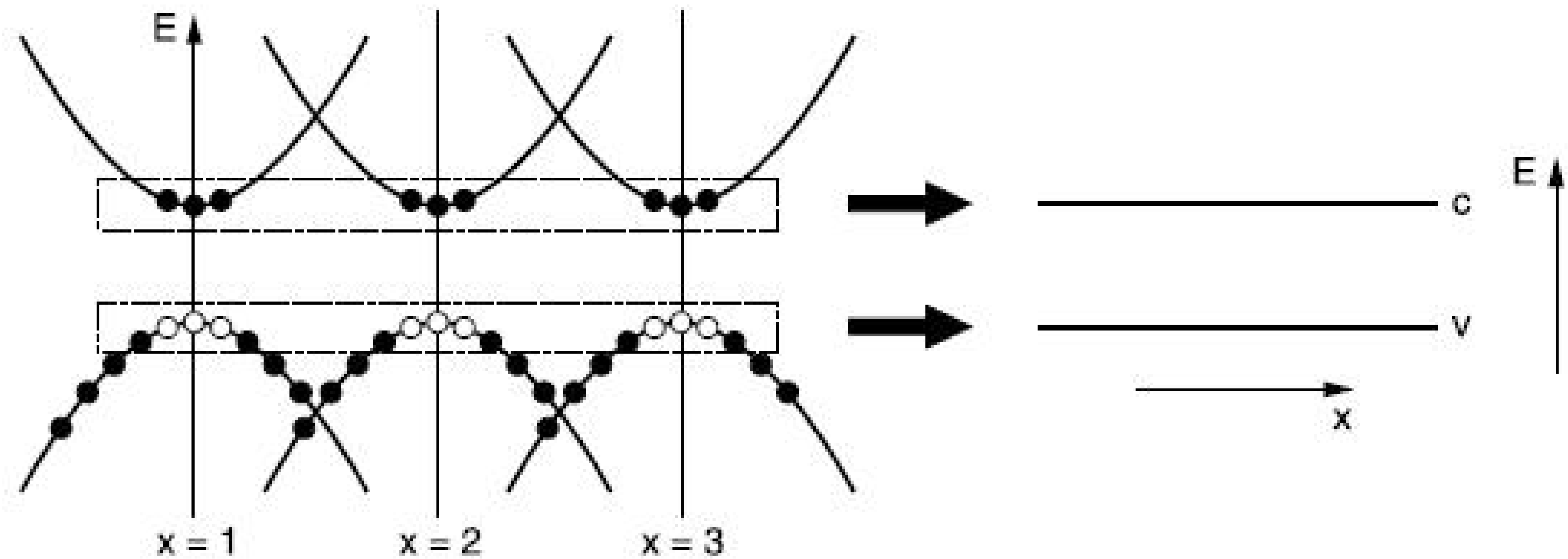


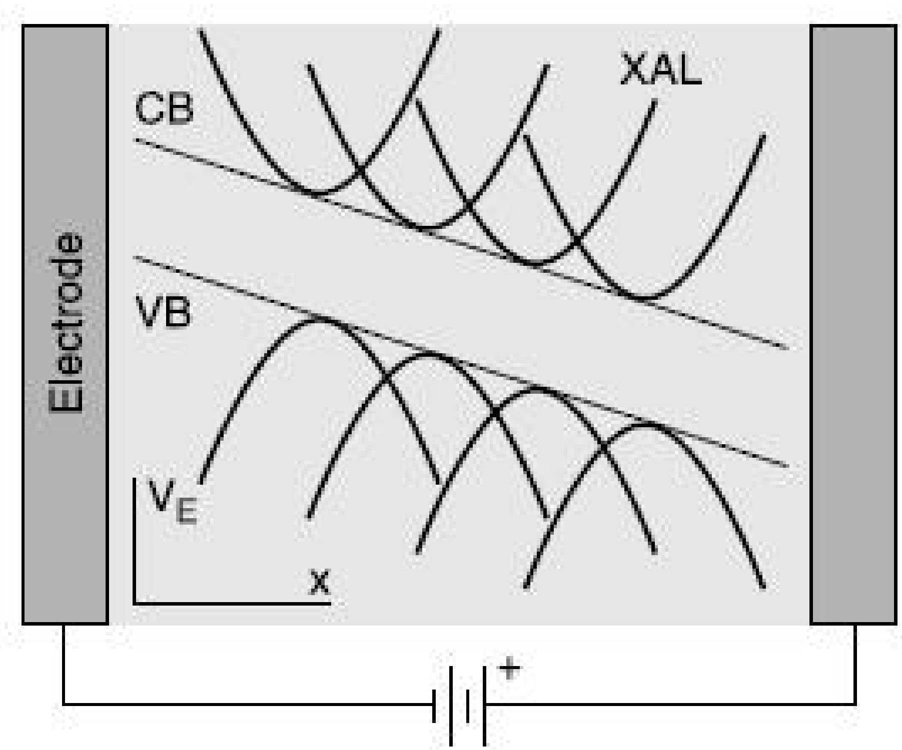
FIGURE 1.5.6
GaAs has a light LH and heavy HH hole valence band.

$$\frac{1}{m_{\text{eff}}} = \frac{1}{\hbar^2}\frac{\partial^2 E}{\partial k^2}$$

Introduction to Band Edge Diagrams




E
x = 1
x = 2
x = 3
E
c
v
x


Electrode
CB
VB
XAL
V_E
x
+

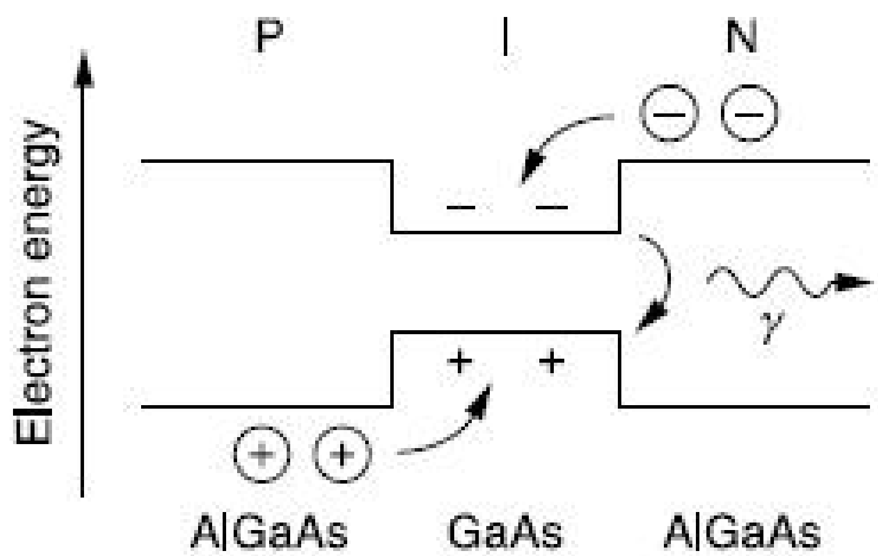
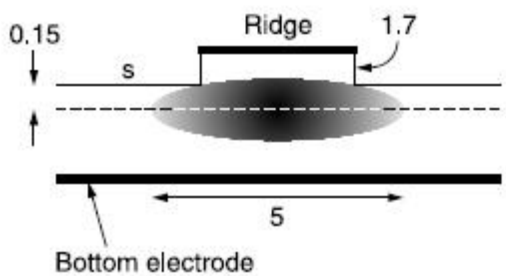
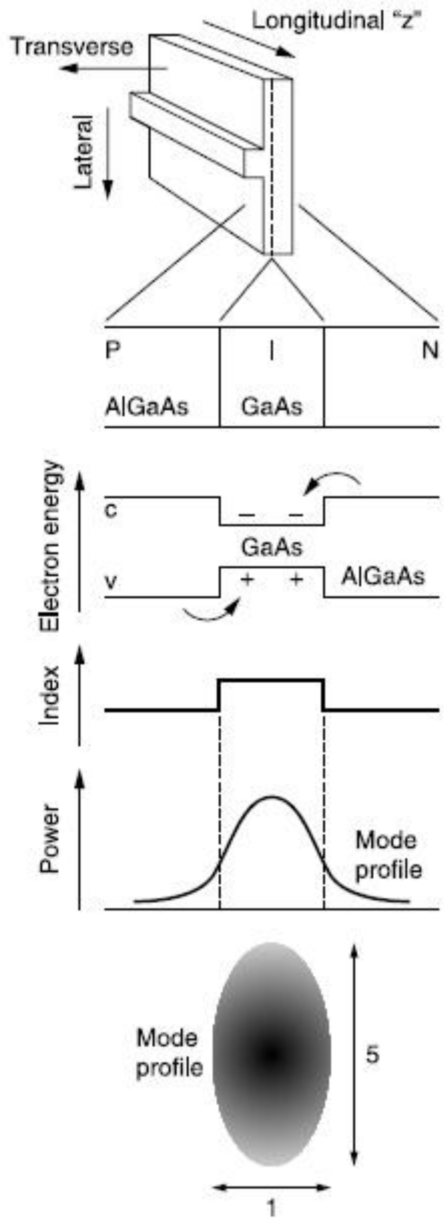
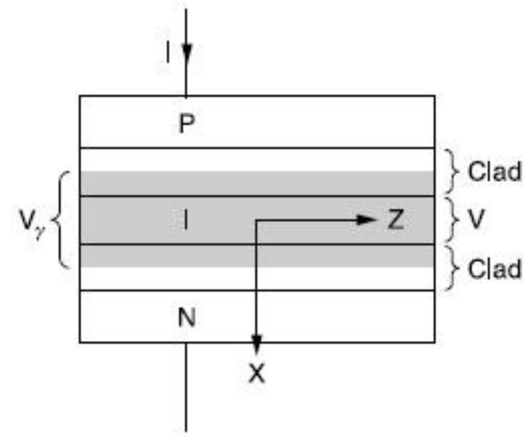


FIGURE 1.5.10
Band-edge diagram for heterostructure with a single quantum well.


0.15
Ridge
1.7
s
5
Bottom electrode


Longitudinal "z"
Transverse
Lateral
P
I
N
AlGaAs
GaAs
Electron energy
c
v
GaAs
AlGaAs
Index
Power
Mode
profile
Mode
profile
5
1


I
P
Clad
I
Z
V
Clad
N
X

Introduction to Laser Dynamics

معادلات ماکسول به کمک مکانیک کوانتم اندر کنش بین فوتون و الکترون را تعیین می کند
به کمک این معادلات می توان معادلات نرخ رسید

The rate equations describe how the gain, pump, feedback, and output couplermechanisms affect the carrier and photon concentration in a device.

معادلات نرخ معین کننده اندرکنش بین فوتون و الکترون در یک محیط با بهره می باشد

The **photon rate equation** describes the effects of the output coupler and feedback mechanism through a relaxation term incorporating the cavity lifetime.

Introduction to the Rate Equations

اندر کنش نور و ماده سبب ایجاد پدیده های متعددی از قبیل گسیل و جذب فوتون و غیره می گردد

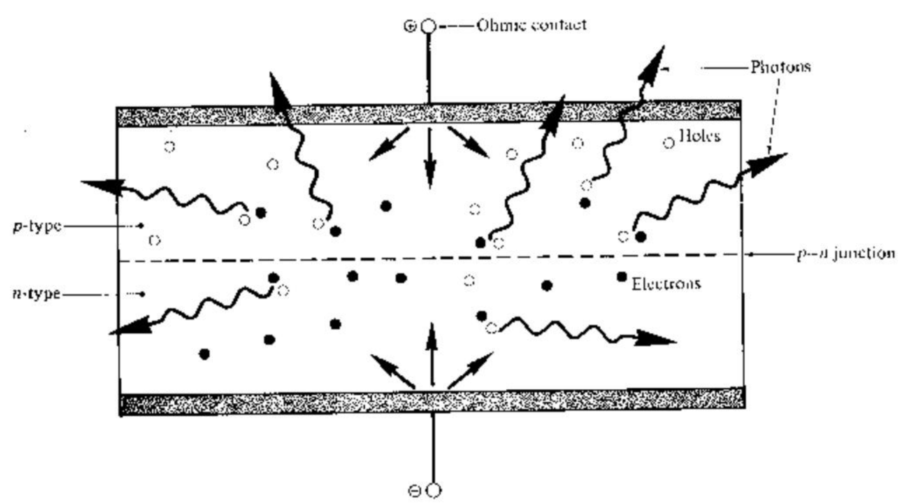


Fig. 6.12 An illustration of carrier recombination giving spontaneous emission of light in a p–n junction diode.

تعدادی از پارامترهایی مورد نیاز:

Density of photons (the number of photons per cm3), γ

the density of electrons $n\ (\#\bar{}/\mathrm{cm}^3)$

pump-current number density $\mathscr{J}\ (\#\mathrm{carriers/s/cm}^3)$.

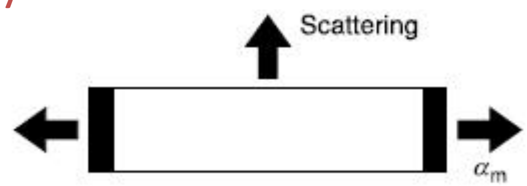
A second set of equations uses variables describing the optical power P (W) and current I (A).

+ SEVERAL SETS of PARAMETERS

$\gamma = \frac{\#\,\text{photons}}{cm^3}$	P = WATTS	$\vec{E}$ = Electric Field
	I = currents	ϕ = phase
$n = \frac{\#\,e^-}{cm^3}$, J	Q = charge	$n = \frac{\#\,e^-}{vol}$

Basic principle: Rapid change of state of an electron

- Must go from high energy state to lower energy state
- Loss of energy must appear somewhere
  - Emitted in form of light
  - Lattice vibrations
- Electron can be bound to a particular atom or molecule or be free (part of electron "gas"), as in most conductors



Two types of emission of light

- Spontaneous
- Stimulated

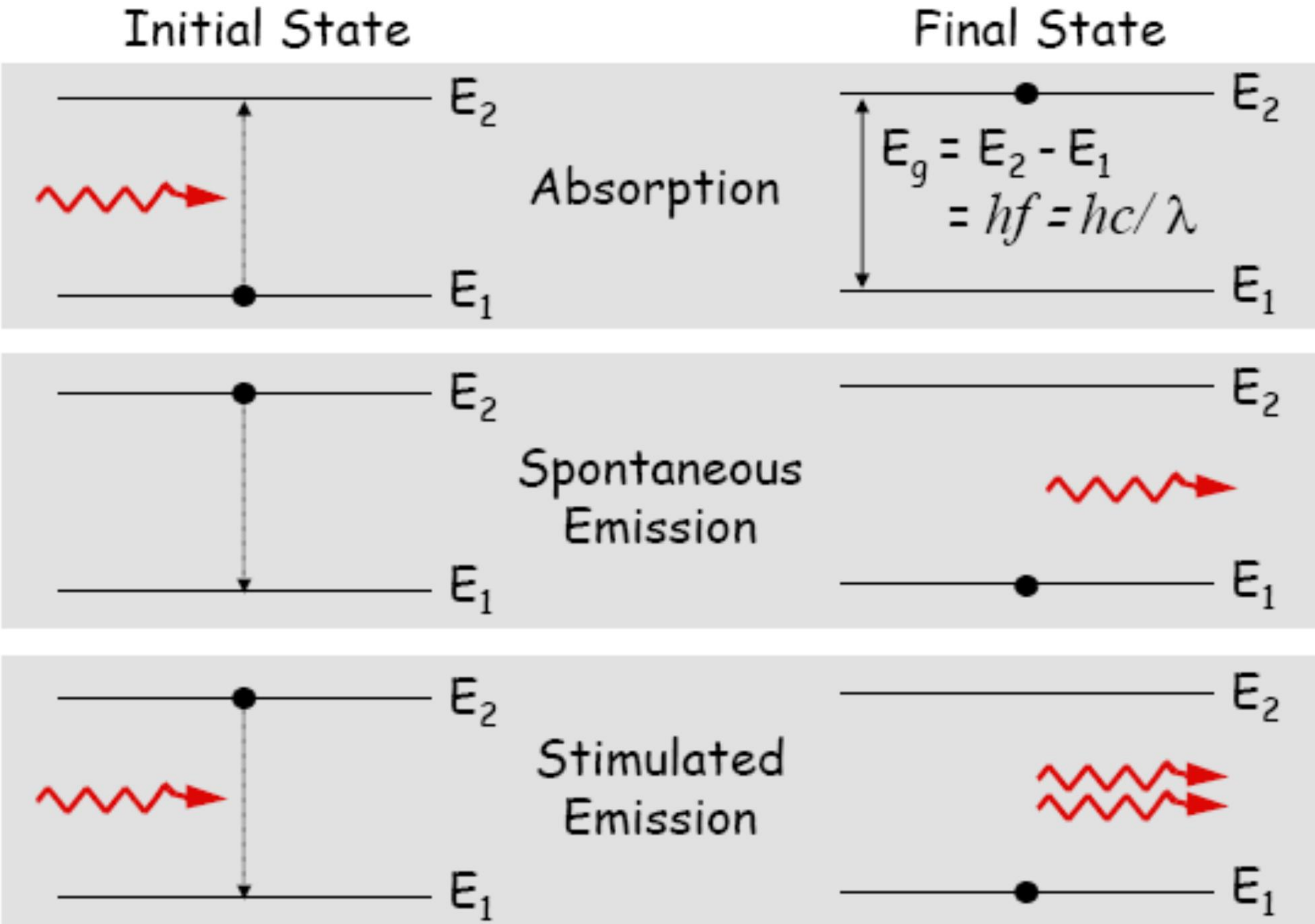
Light production: spontaneous

- Normal case
- Electron in high energy state is unstable
  - Spontaneously returns to lower state
  - Occurs in a few picoseconds
  - Photon emitted in process
  - Direction, phase random
  - Energy of photon determined by transition undergone by electron

Light production: stimulated

- Electron in high energy metastable state
  - Can remain there for a “long” time (~microseconds)
  - Can fall back spontaneously, or be “stimulated” to emit its energy by another photon
    - Incident photon must have correct energy
    - Newly emitted photon will have same wavelength, phase, and direction as incident (stimulating) photon

ABSORPTION & EMISSION



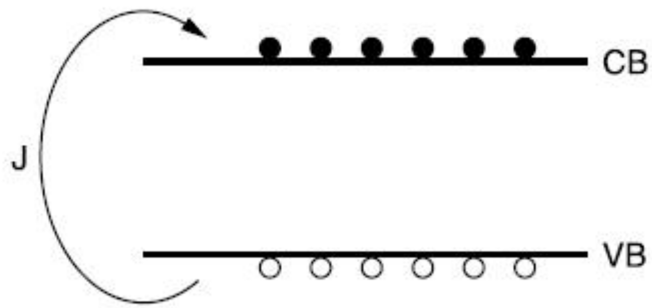
The Simplest Rate Equations

مشخص کننده تغییر زوج حاملها در درون ناحیه فعال یک لیزر

$$\frac{dN}{dt} = \text{Generation-Recombination}$$

$$N = nV$$

We assume
- intrinsic semiconductor (n=p)
- two levels



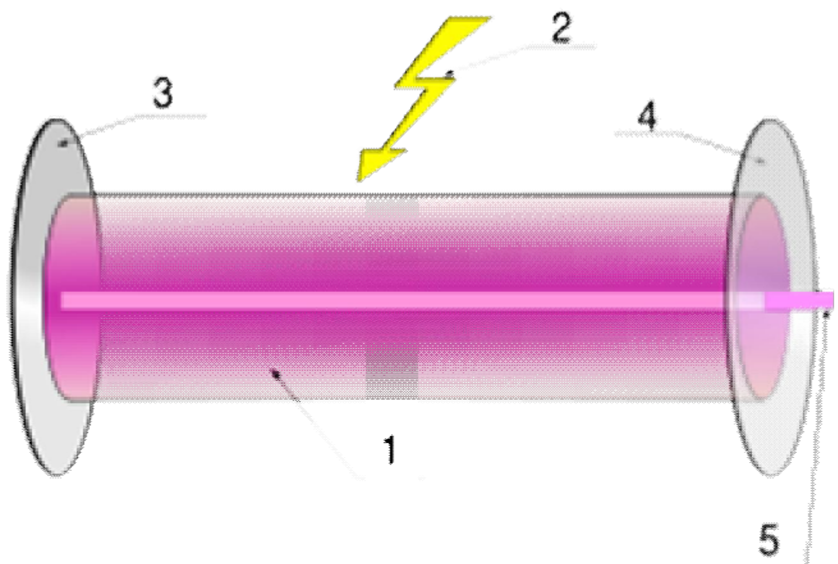
Generation
- pumping
- absorption

Recombination
- stimulated
- spontaneous

$$\frac{dN}{dt} = -\begin{pmatrix}\text{Stimulated}\\ \text{Emission}\end{pmatrix} + \begin{pmatrix}\text{Stimulated}\\ \text{Absorption}\end{pmatrix} + \text{Pump} - \begin{pmatrix}\text{Non-Radiative}\\ \text{Recombination}\end{pmatrix} - \begin{pmatrix}\text{Spontaneous}\\ \text{Recombination}\end{pmatrix}$$

پمپ توسط جریانی که از بالایی لیزر به آن تزریق می شود و یا از طریق یک شدت نور ایجاد می گردد

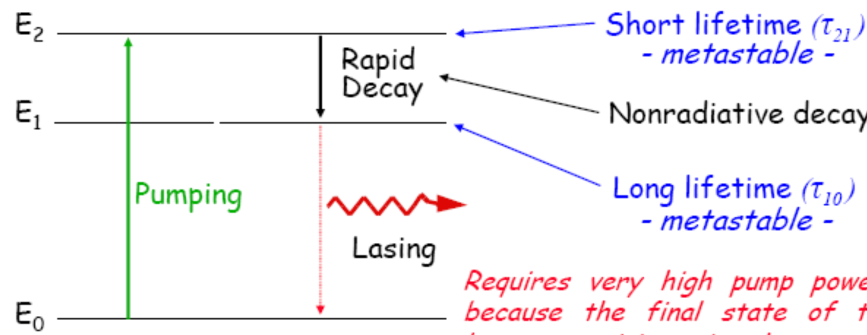
$$\text{Pump} = \mathscr{J}V$$



افزایش یا کاهش الکترونها نسبت عکس با تعداد فوتونها دارد

ENERGY LEVEL DIAGRAM

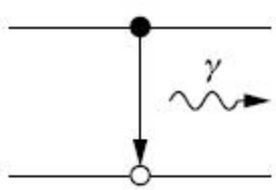
- Three Level System -



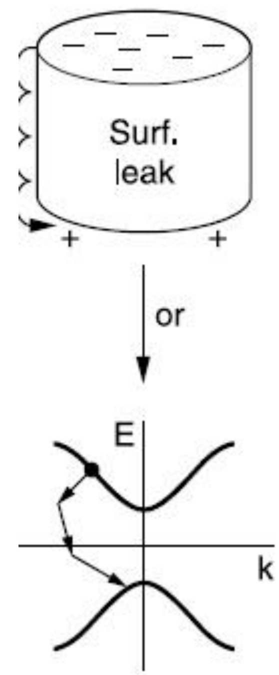
Three Level System
Ruby Laser (crystal)

Requires very high pump powers because the final state of the laser transitions is the ground state. Thus, over half of the ground state atoms must be pumped into the metastable state to achieve population inversion

Spontaneous absorption

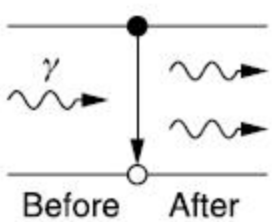


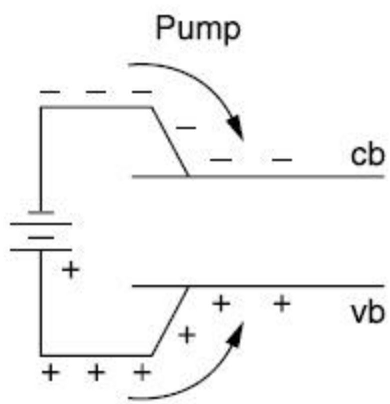
Non-radiative recombination



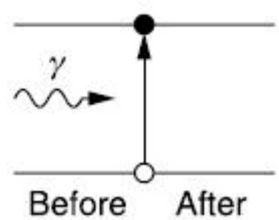
$$\frac{d\Upsilon}{dt} = +\begin{pmatrix}\text{Stimulated}\\ \text{Emission}\end{pmatrix} - \begin{pmatrix}\text{Stimulated}\\ \text{Absorption}\end{pmatrix} - \begin{pmatrix}\text{Optical}\\ \text{Loss}\end{pmatrix} + \begin{pmatrix}\text{Fraction of}\\ \text{Spont. Emiss.}\end{pmatrix}$$

Stimulated recombination

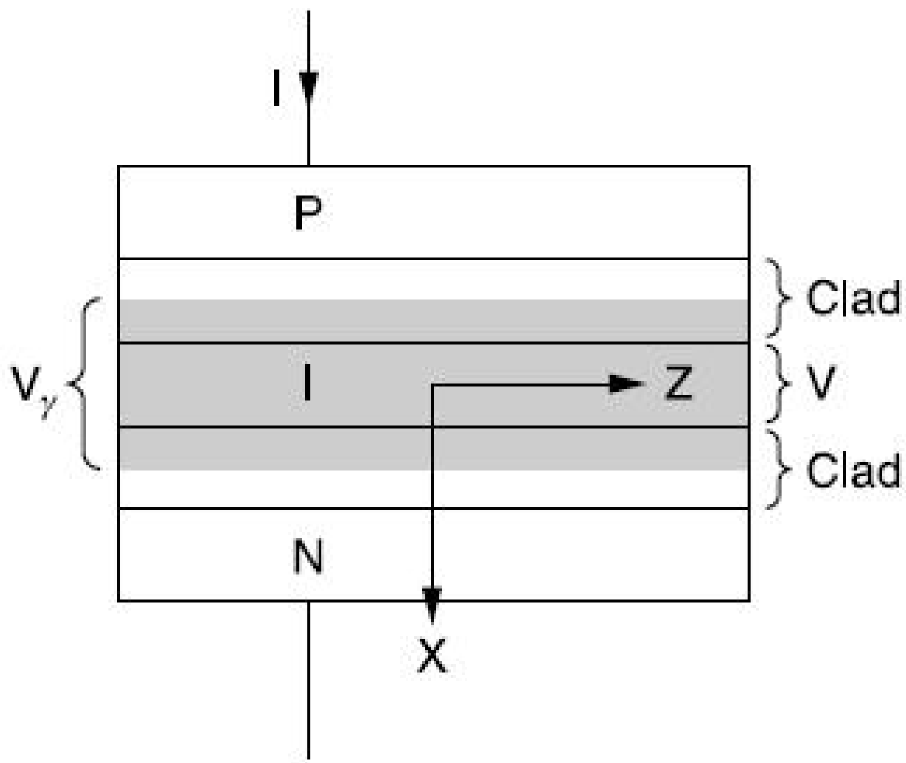




Stimulated absorption



Optical Confinement Factor



$$\Gamma = V / V_{\gamma}$$

The Pump Term and the Internal Quantum Efficiency

The number of electron–hole pairs that contribute to the **photon emission** process in each unit of volume (cm3) of the active region in **each second** can be related to the **bias current** I by

Internal Quantum Efficiency

$$\mathscr{J} = \frac{\eta_i I}{qV}$$

represents the fraction of terminal current I that generates carriers in the active region

where $\mathscr{J}$ represent the "pump-current number density,

$\eta_i I$ → **provides the actual current absorbed in the active region**

Pumping (whether optical or electrical) initiates laser action when the carrier population reaches the ‘‘threshold’’ density (to be discussed later).

Without carrier recombination, the pump would continuously increase the carrier population according to

$$dn/dt = \mathcal{J} \ (\text{recall } n = p).$$

جهت جلوگیری از پراکندگی بحث هدف این است که روابط زیر را بدست آوریم (معادلات نرخ حاملها و موج)

$$V\frac{dn}{dt} = -\begin{pmatrix}\text{Stimulated}\\ \text{Emission}\end{pmatrix} + \begin{pmatrix}\text{Stimulated}\\ \text{Absorption}\end{pmatrix} + JV - \begin{pmatrix}\text{Non-Radiative}\\ \text{Recombination}\end{pmatrix} - \begin{pmatrix}\text{Spontaneous}\\ \text{Recombination}\end{pmatrix}$$

$$V\frac{dn}{dt} = -V\,v_g g\,\gamma + JV - \frac{n}{\tau_e}V$$

$$\frac{1}{\tau_e} = A + Bn + Cn^2 \qquad \text{and} \qquad A = 1/\tau_n$$

For good material, $A = C = 0$ $$\frac{dn}{dt} = -\,v_g g\,\gamma + J - Bn^2$$

$$V_\gamma \frac{d\gamma}{dt} = +\begin{pmatrix}\text{Stimulated}\\ \text{Emission}\end{pmatrix} - \begin{pmatrix}\text{Stimulated}\\ \text{Absorptions}\end{pmatrix} - \begin{pmatrix}\text{Optical}\\ \text{Loss}\end{pmatrix} + \begin{pmatrix}\text{Fraction of}\\ \text{Spont. Emiss.}\end{pmatrix}$$

$$V_\gamma \frac{d\gamma}{dt} = +V\, v_g\, g(n)\,\gamma - V_\gamma \frac{\gamma}{\tau_\gamma} + \beta B n^2 V_\gamma$$

$$\frac{d\gamma}{dt} = +\Gamma\, v_g\, g(n)\,\gamma - \frac{\gamma}{\tau_\gamma} + \beta B n^2$$

$$\frac{dn}{dt} = -v_g\, g(n)\,\gamma + J - Bn^2$$

$$\frac{d\gamma}{dt} = +\Gamma v_g\, g(n)\,\gamma - \frac{\gamma}{\tau_\gamma} + \beta B n^2$$

که حال همگی این پارامترها توضیح داده خواهد شد

Recombination Terms
- radiative
- nonradiative

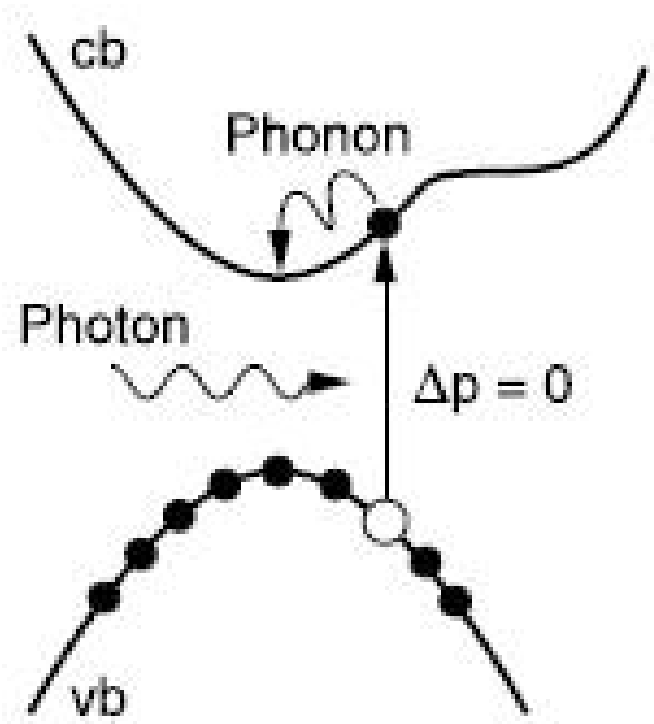
radiative

produces photons usually as spontaneous emission

Nonradiative recombination

occurs primarily through **phonon p**rocesses by material with **indirect bandgap**

The result is heat



Main process in the Laser

Monomolecular (nonradiative) $\frac{dn}{dt} = -\frac{n}{\tau_n} = -An$ $R_{\text{mono}}V = AnV$

bimolecular (radiative)

produces spontaneous emission $R_{sp}V = Bn^2V$

Auger recombination (nonradiative)

occurs when carriers transfer their energy to other carriers, which interact with phonons to return to an equilibrium condition

Auger recombination **is important** for lasers (such as InGaAsP) with emission wavelengths **larger than 1 mm (small bandgap)**

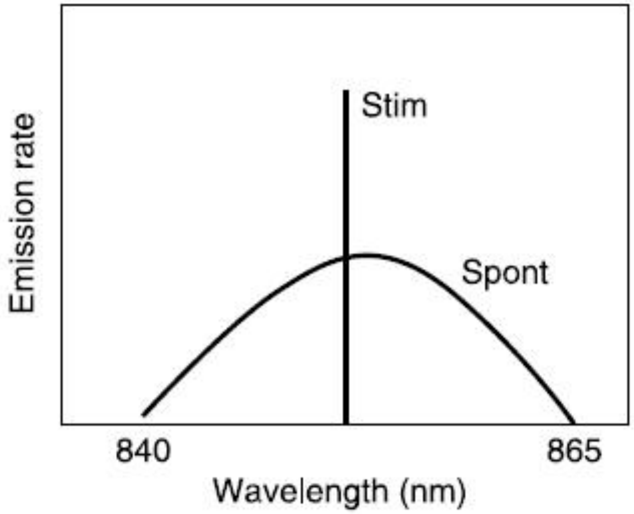
$$R_{\text{aug}}V = Cn^3V$$

$$R_r = R_{\text{radiative}} + R_{\text{nonradiative}} = An + Bn^2 + Cn^3$$

$$\frac{1}{\tau_e} = A + Bn + Cn^2$$

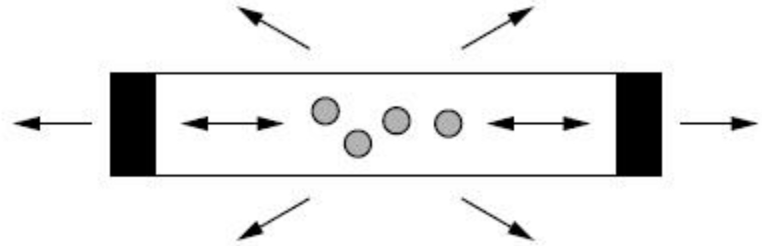
Spontaneous Emission Term

Of those photons that enter the waveguide, a fraction of them have exactly the right frequency to match that of the lasing mode



$$V_\gamma R_{sp} = V_\gamma \beta B n^2$$

The Optical Loss Term



All of the optical losses contribute to an overall relaxation time τ_γ

called the cavity lifetime

$$V_\gamma \frac{d\gamma}{dt} = -\frac{V_\gamma \gamma}{\tau_\gamma}$$

$$\gamma(t) = \gamma_o \exp\left(-\frac{t}{\tau_\gamma}\right)$$

which shows that the initial photon density decays exponentially as the carriers are lost

Absorption and Gain

$$\alpha = \tau_\gamma^{-1} / v_g$$

$$\frac{1}{\tau_\gamma} = \alpha v_g$$

v_g represents the group velocity of the wave.

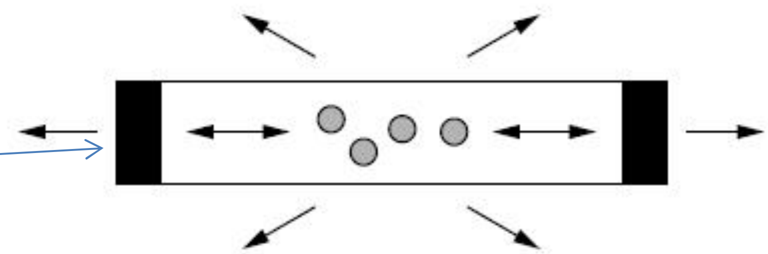
mirror loss α_m

as distributed along the length of the cavity

a partial differential equation with **boundary conditions** at the mirrors must be solved

result

$$\frac{1}{\tau_m} = v_g \alpha_m = \frac{v_g}{L} Ln(1/R)$$



both mirrors have the same power reflectivity R (0.34 for GaAs).

The loss per mirror must be $\alpha_m/2.$

reciprocal of the cavity lifetime becomes

$$\frac{1}{\tau_\gamma} = \frac{1}{\tau_{\text{int}}} + \frac{1}{\tau_m} = v_g \alpha = v_g(\alpha_{\text{int}} + \alpha_m)$$

Stimulated Emission—Absorption and Gain

types of gain
- Temporal Gain
- Single Pass Gain
- Material Gain
- Material Transparency
- Introduction to the Energy Dependence of Gain

Temporal Gain

The CB electrons and VB holes produce "gain" in the sense that incident photons with the proper wavelength can stimulate **carrier recombination** and thereby produce **more photons** with the same characteristics as the incident ones.

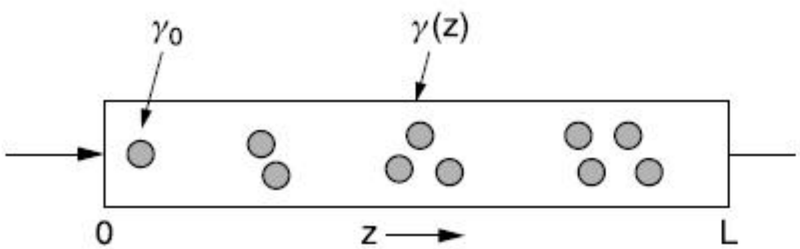
The word ''stimulated'' means that a photon must be incident on the material before either stimulated emission or absorption can proceed

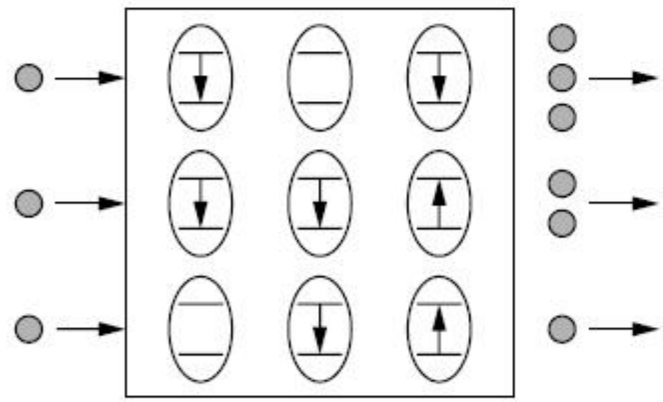
$$R_{\text{stim}} V_\gamma = V_\gamma \frac{d\gamma}{dt}\bigg|_{\text{stim}} \sim \gamma$$

$$R_{\text{stim}} V_\gamma = V_\gamma \frac{d\gamma}{dt}\bigg|_{\text{stim}} = V g_t \gamma$$

$$R_{\text{stim}} = \frac{d\gamma}{dt}\bigg|_{\text{stim}} = \frac{V}{V_\gamma} g_t \gamma = \Gamma g_t \gamma$$

Single Pass Gain





$$\frac{d\gamma}{dz} = \Gamma g \gamma \quad \rightarrow \quad \gamma(z) = \gamma(0)\, e^{\Gamma g z}$$

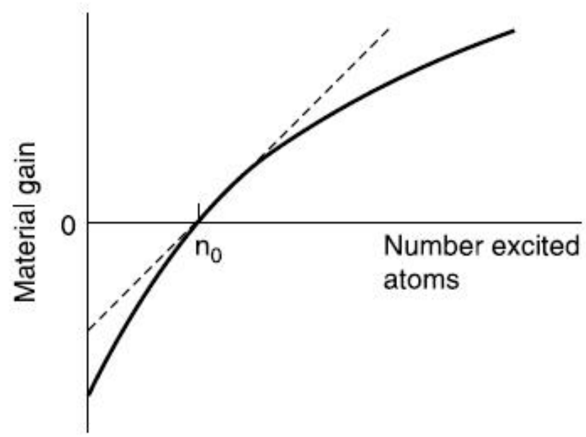
$$G = e^{\Gamma g z}$$

Material Gain

$$g \sim \frac{1}{\text{Length}} \sim \frac{1}{\text{Sec}}\frac{\text{Sec}}{\text{Length}} \sim g_t \frac{t}{L} \sim \frac{g_t}{v_g}$$

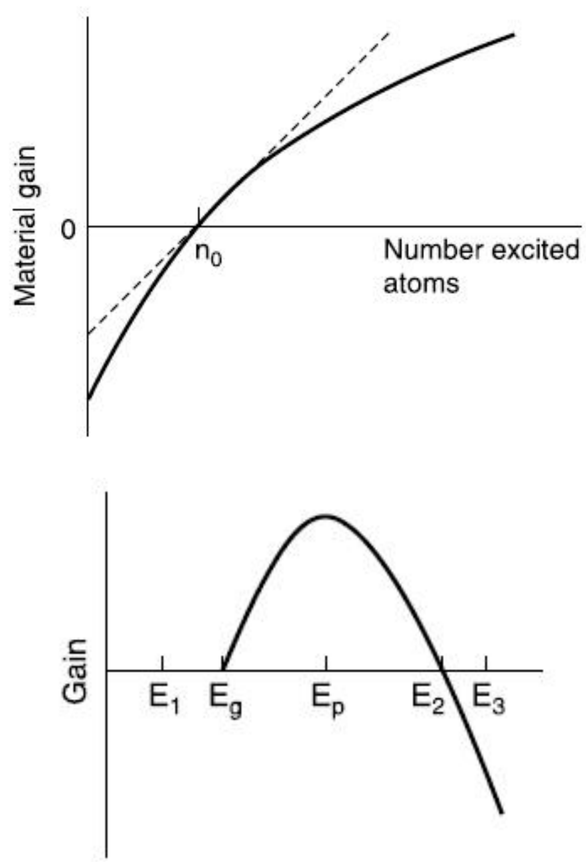
$$g(n) = g_o \,\text{Ln}\left(\frac{n - n_\infty}{n_o - n_\infty}\right)$$

$$g_o(n) = \frac{dg(n)}{dn}$$

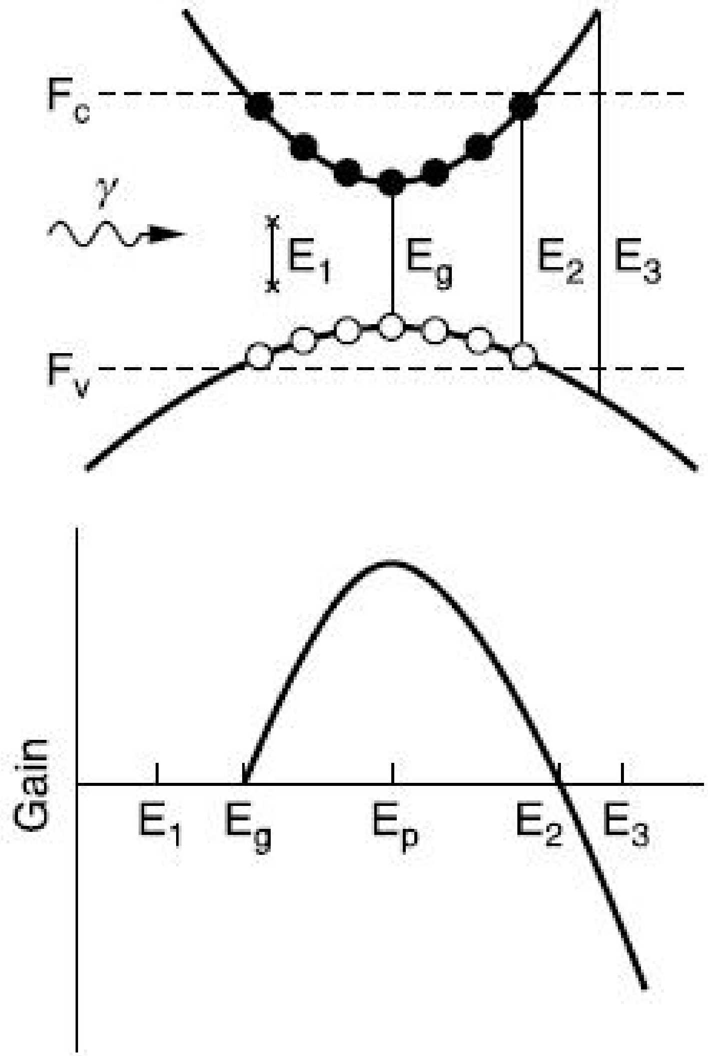


Material Transparency

G=1. The material gain in this case is g=0

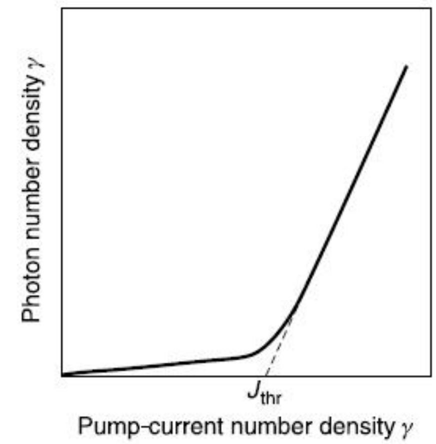


Introduction to the Energy Dependence of Gain



The Power–Current Curves

P–I or *L–I* curves.



Photon Density versus Pump-Current Number Density

$$\frac{dn}{dt} = -v_g\, g(n)\, \gamma + \mathscr{J} - Bn^2$$

$$\frac{d\gamma}{dt} = +\Gamma v_g\, g(n)\, \gamma - \frac{\gamma}{\tau_\gamma} + \beta B n^2$$

steady state

$$0 = -v_g g(n)\gamma + \mathscr{J} - Bn^2$$

$$0 = +\Gamma v_g g(n)\gamma - \frac{\gamma}{\tau_\gamma} + \beta Bn^2$$

Below Lasing Threshold

The phrase ''below lasing threshold'' implies that the laser has insufficient gain to support oscillation

$$\mathscr{J} < \mathscr{J}_{\text{thr}}$$

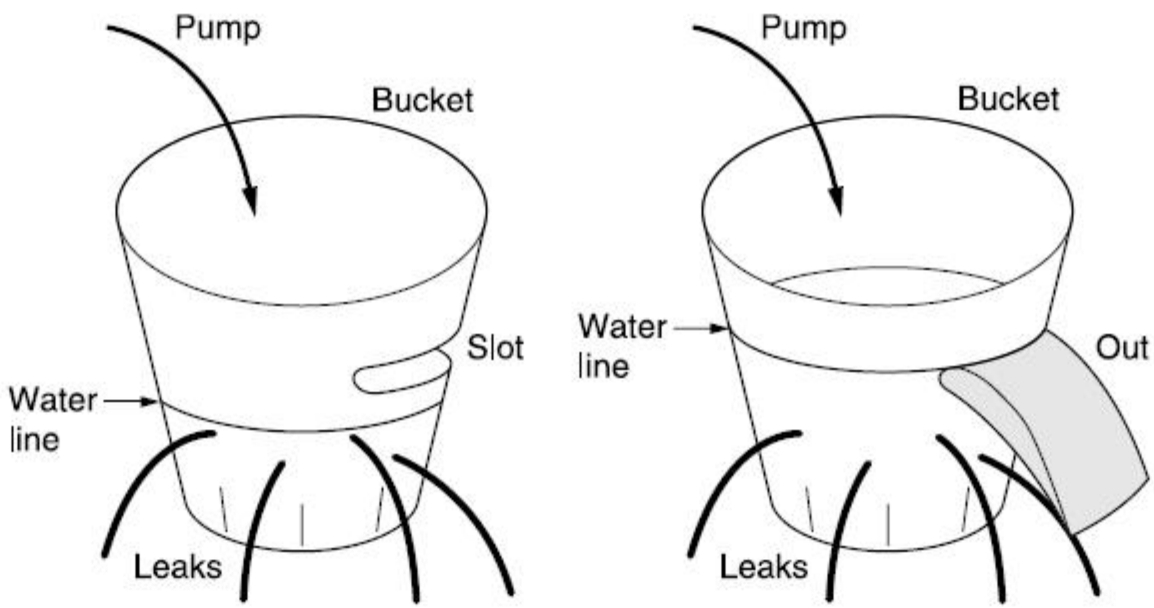
در حالت پایدار

$$g=0$$

$$\gamma = \beta \tau_\gamma B n^2$$

$$Bn^2 = \mathscr{J}$$

$$\gamma = \beta \tau_\gamma \mathscr{J}$$



$$\gamma = \beta \tau_\gamma B n^2$$

$$B n^2 = \mathscr{J}$$

$$\gamma = \beta \tau_\gamma \mathscr{J}$$

Above Lasing Threshold $(J > J_{thr})$

$$0 = -v_g g(n)\gamma + \mathscr{J} - R_{rec}$$

$$0 = +\Gamma v_g g(n)\gamma - \frac{\gamma}{\tau_\gamma}$$

$$R_{rec} = \frac{n}{\tau_e} = \frac{n}{\tau_n} + Bn^2 + Cn^3 \cong Bn^2$$

$$\frac{\text{Power out}}{\text{two mirrors}} = P_o = \frac{\text{Energy}}{\text{Sec}} = \left(\frac{\text{Energy}}{\text{Photon}} * \frac{\text{Photons}}{\text{Volume}} * \text{Mode volume}\right) * \frac{1}{\tau_m}$$

mirror time constant is

$$1/\tau_m = v_g \alpha_m.$$

energy of each photon

photons striking a single mirror must be proportional to γ/2 and for both mirrors

modal volume

$$P_o = \gamma \frac{hc}{\lambda_o} V_\gamma v_g \alpha_m$$

wavelength in vacuum

Case 1 P–I Below Threshold

We have:

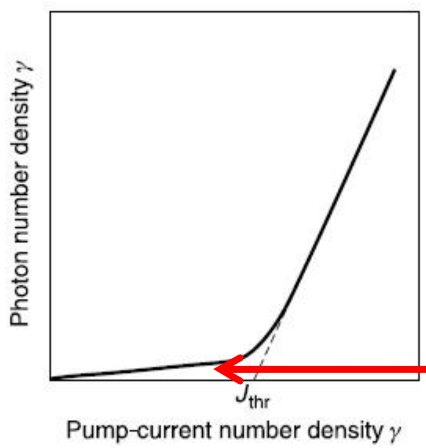
$$\gamma = \beta \tau_\gamma \mathscr{J}$$

$$P_o = \gamma \frac{hc}{\lambda_o} V_\gamma v_g \alpha_m$$

$$P_o = (\beta \tau_\gamma \mathscr{J}) \frac{hc}{\lambda_o} V_\gamma v_g \alpha_m$$

Next life time is:

$$\frac{1}{\tau_\gamma} = \frac{1}{\tau_m} + \frac{1}{\tau_{\text{int}}} = v_g(\alpha_m + \alpha_{\text{int}})$$



$$P_o = \beta \tau_\gamma \frac{\eta_i I}{qV} \frac{hc}{\lambda_o} V_\gamma v_g \alpha_m = \beta \frac{\eta_i}{q\Gamma} \frac{hc}{\lambda_o} \frac{\alpha_m}{\alpha_m + \alpha_{\text{int}}} I$$

Case 2 P–I Above Threshold

$$I > I_{\text{thr}}$$

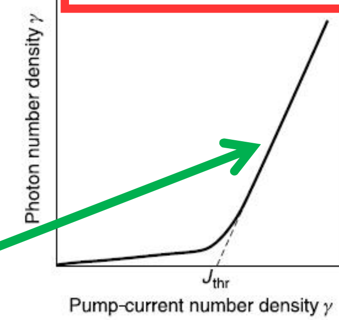
$$P_o = \gamma \frac{hc}{\lambda_o} V_\gamma v_g \alpha_m$$

$$J = \eta_i I / (qV),$$

$$\gamma = \Gamma \tau_\gamma (\mathscr{J} - \mathscr{J}_{\text{thr}})$$

$$\frac{P_o}{(hc/\lambda_o)\gamma V_\gamma v_g \alpha_m} = \Gamma \tau_\gamma \left(\frac{\eta_i I}{qV} - \frac{\eta_i I_{\text{thr}}}{qV} \right)$$

$$I_{\text{thr}} = (qV/\eta_i)\mathscr{J}_{\text{thr}}$$



با در نظر گرفتن موارد زیر:

$$\Gamma = V/V_\gamma$$

$$\frac{1}{\tau_\gamma} = \frac{1}{\tau_m} + \frac{1}{\tau_{\text{int}}} = v_g(\alpha_m + \alpha_{\text{int}})$$

$$P_o = \eta_i \frac{hc}{q\lambda_o} \frac{\alpha_m}{\alpha_{\text{int}} + \alpha_m} (I - I_{\text{thr}})$$

Power versus Voltage

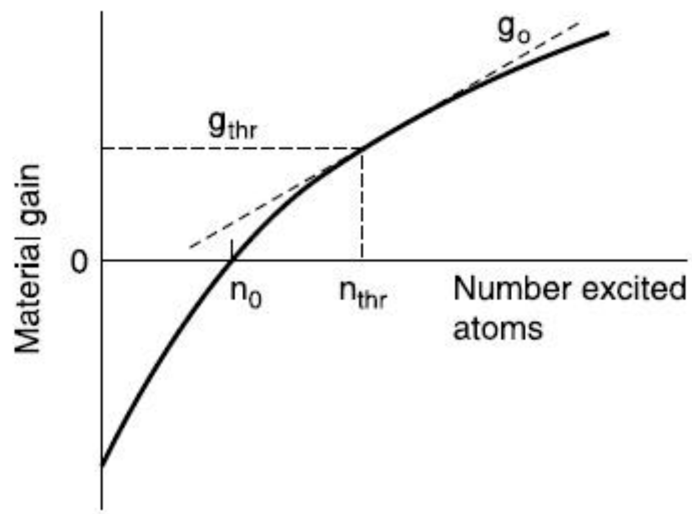
$$I = I_o\left(e^{qV/kT} - 1\right) \sim I_o\, e^{qV/kT}$$

Some Comments on Gain

$$g_{\text{thr}} = g(n_{\text{thr}}) = g_o(n_{\text{thr}} - n_o).$$

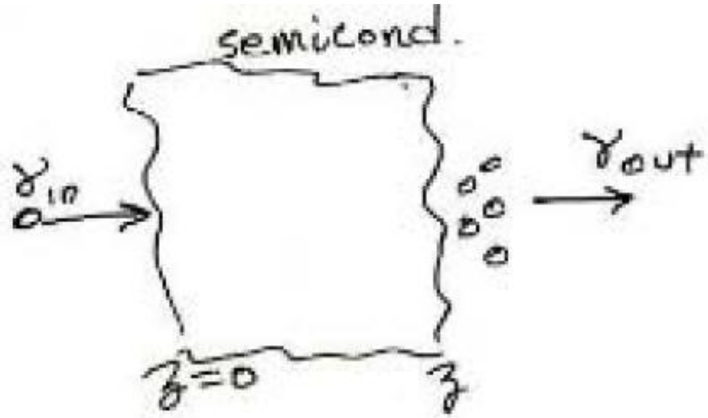
$$g_o(n) = \frac{dg(n)}{dn}$$

$$g(n) = g_o(n - n_o)$$



Material trancparency

.ر نظر گرفتن انعکاس و پراکندگی و بدون در نظر گرفتن آینه ها



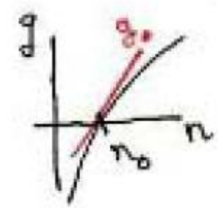
$$\frac{d\gamma}{dt} \simeq v_g g \gamma \Rightarrow \frac{d\gamma}{dz} = g\gamma$$

$$\Rightarrow \gamma_{out} = \gamma_{in} e^{gz}$$

Trancparency occur when $\gamma_{out} = \gamma_{in}$

$$1 = \frac{\gamma_{out}}{\gamma_{in}} = \underbrace{G}_{\text{single pass}} = e^{gz} \rightarrow g \simeq 0$$

$$n = n_0 = \text{transp. carr. density}$$

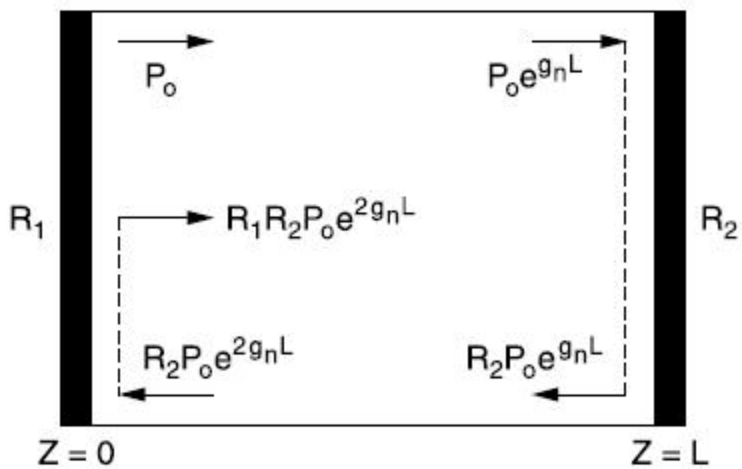


$$g(n) \simeq g_0 (n - n_0)$$

$$g_0 = \frac{dg}{dn} = \text{different. gain}$$

Relations for Cavity Lifetime, Reflectances and Internal Loss

mirror mirror



R=power Reflectances

Lock at round trip

Relation between P1 and P2 with only material gain and internal loss

$$\frac{d\gamma}{dt} = +\Gamma v_g g(n)\gamma - \frac{\gamma}{\tau_\gamma}$$

internal losses

$$\frac{1}{\tau_\gamma} = v_g\alpha_{\text{int}} + v_g\alpha_m = v_g\alpha_{\text{int}} + \frac{v_g}{2L}\text{Ln}\left(\frac{1}{R_1R_2}\right)$$

$$\frac{1}{\tau_\gamma} = \frac{1}{\tau_{\text{int}}} = v_g\alpha_{\text{int}}$$

the cavity lifetime

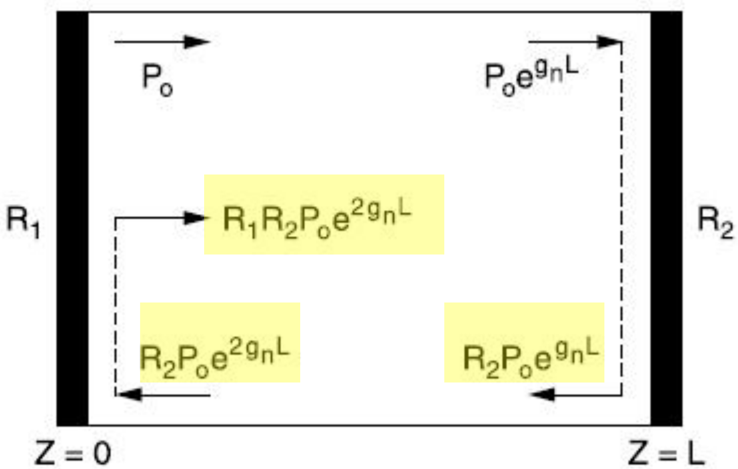
No reflection facet $a_{m=0}$

$$\frac{d\gamma}{dt} = v_g \frac{d\gamma}{dz} = v_g g\gamma - \gamma v_g \alpha_{\mathrm{int}} \quad \text{or} \quad \frac{d\gamma}{dz} = g\gamma - \gamma\alpha_{\mathrm{int}} = g_{\mathrm{net}}\gamma$$

where the net gain $g_{\mathrm{net}} = g - \alpha_{\mathrm{int}}$

So we have $\frac{dP}{dz} = g_{\mathrm{net}} P \Rightarrow P(z) = P_o \exp(g_n z)$

from $z=0$ to $z=L$.



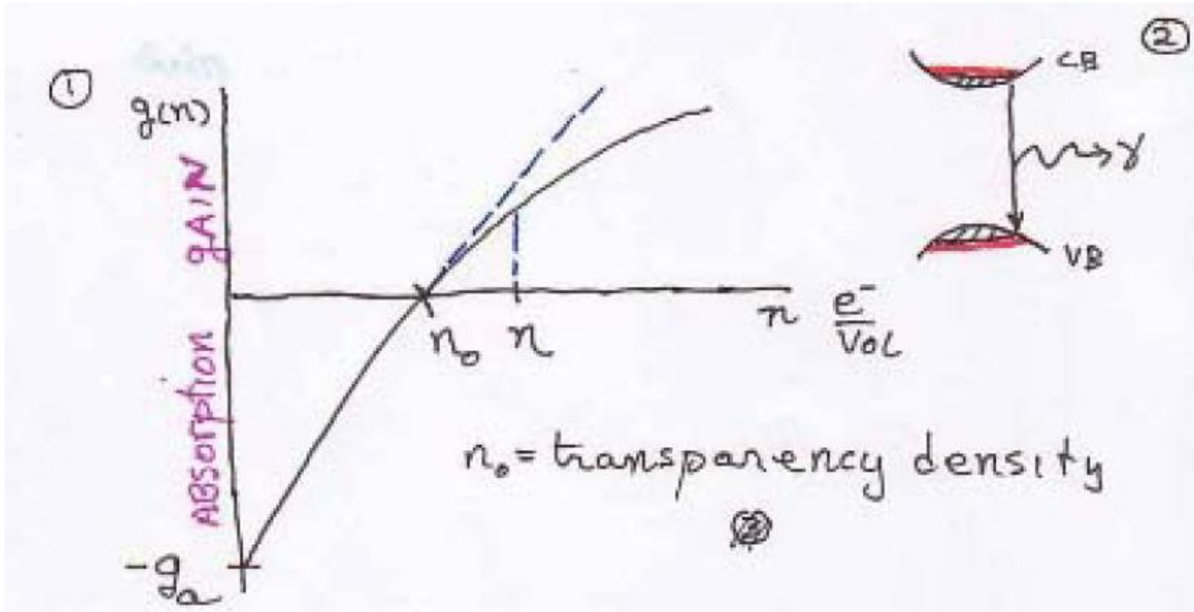
For steady state the initial power Po must be the same as the final power

$$P_o = R_1 R_2 P_o \exp(2 g_n L)$$

$$g_{\text{net}} = \frac{1}{2L} \text{Ln}\left(\frac{1}{R_1 R_2}\right)$$

which says that the gain must equal the loss at steady state.

Some BASIC FACTS About Gain



- Recall $\gamma \sim e^{gL}\gamma_0$ g = MATERIAL only

  $g = 0 \rightarrow \gamma = \gamma_0$ NOT $g_{net} = g - \alpha_{int}$

  TRANSPARENCY occurs at n_0

  1 photon 'in' → 1 photon 'out'

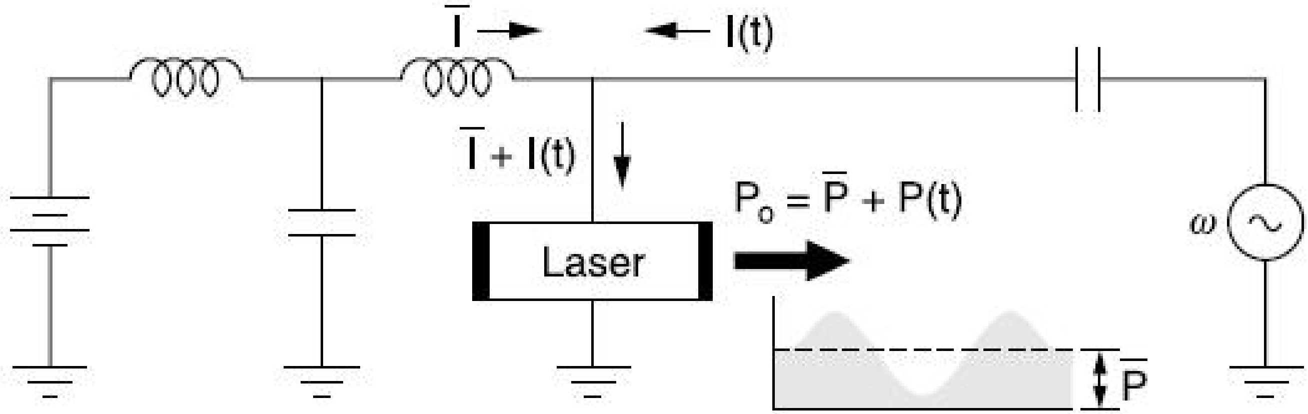
- Can linearize $g = g(n)$ by Taylor expanding about n_0

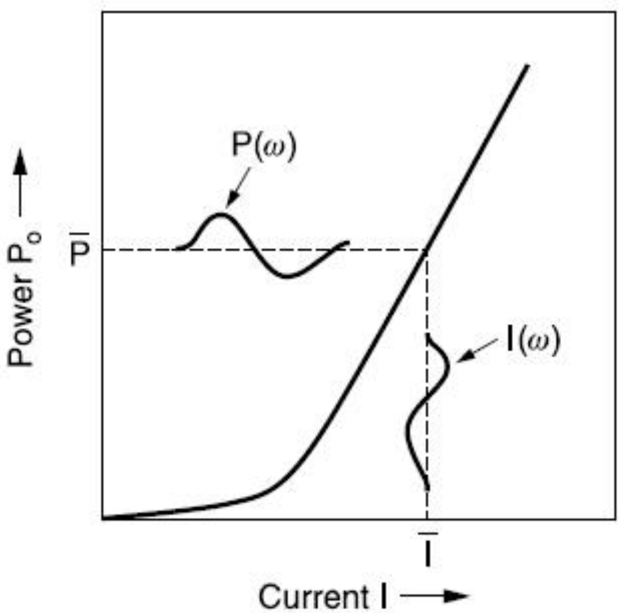
$$g \sim g_0'(n - n_0)$$

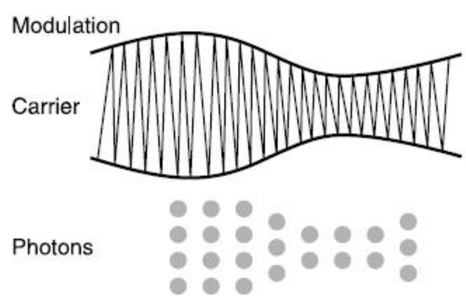
- $n < n_0 \rightarrow g < 0 \rightarrow$ $\gamma = e^{gL}\gamma_0 < \gamma_0$ Absorp
- $n > n_0 \rightarrow g > 0 \rightarrow$ $\gamma = e^{gL}\gamma_0 > \gamma_0$ Gain
- $n = n_0 \rightarrow g = 0 \rightarrow$ $\gamma = \gamma_0$ Transparency

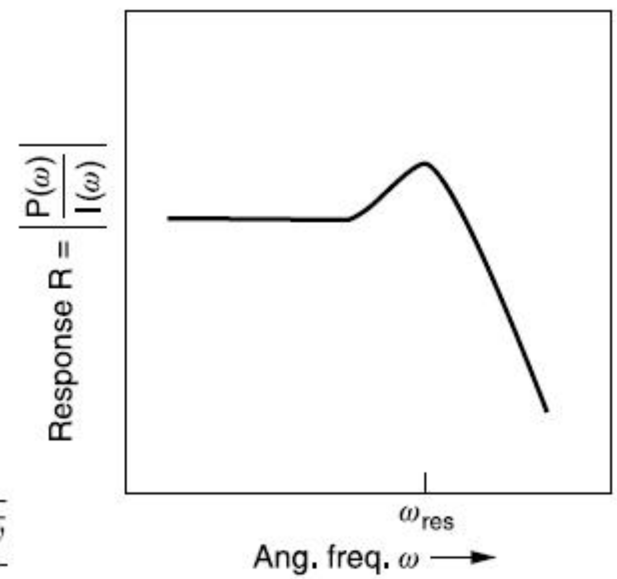
Modulation Bandwidth

$$I(t) = I\,e^{i\omega t}$$









$$\omega_{res} = \sqrt{\frac{v_g g_o \bar{\gamma}}{\tau_\gamma}}$$

By: Dr. Jabbari

Small Signal Analysis

To demonstrate the equation for resonance

"bar' → "average'

Rate and Wave Equation

$$\frac{dn'}{dt} = -v_g g_o (n' - n_o)\gamma' + J' - \frac{n'}{\tau_n}$$

$$\frac{d\gamma'}{dt} = +\Gamma v_g g_o (n' - n_o)\gamma' - \frac{\gamma'}{\tau_\gamma}$$

$$n' = \bar{n} + n\, e^{i\omega t} \qquad \gamma' = \bar{\gamma} + \gamma\, e^{i\omega t} \qquad J' = \bar{J} + J\, e^{i\omega t}$$

For steady-state

$$0 = -v_g g_o(\bar{n} - n_o)\bar{\gamma} + \bar{J} - \frac{\bar{n}}{\tau_n}$$

$$0 = +\Gamma v_g\, g_o(\bar{n} - n_o)\,\bar{\gamma} - \frac{\bar{\gamma}}{\tau_\gamma}$$

$$i\omega n\, e^{i\omega t} = -v_g\left(\bar{g} + g_o n\, e^{i\omega t}\right)\left(\bar{\gamma} + \gamma\, e^{i\omega t}\right) + \bar{J} - \frac{\bar{n}}{\tau_n} + J\, e^{i\omega t} - \frac{n}{\tau_n} e^{i\omega t}$$

$$i\omega\gamma\, e^{i\omega t} = \Gamma v_g\left(\bar{g} + g_o n\, e^{i\omega t}\right)\left(\bar{\gamma} + \gamma\, e^{i\omega t}\right) - \frac{\gamma\, e^{i\omega t}}{\tau_\gamma} - \frac{\bar{\gamma}}{\tau_\gamma}$$

$$i\omega n\, e^{i\omega t} = -v_g\left[g_o n\, e^{i\omega t}\;\bar{\gamma} + \bar{g}\gamma\, e^{i\omega t} + g_o\gamma n\, e^{i\omega t}\right] + J\, e^{i\omega t} - \frac{n}{\tau_n} e^{i\omega t}$$

$$i\omega\gamma\, e^{i\omega t} = \Gamma v_g\left[\bar{\gamma} g_o n\, e^{i\omega t} + \bar{g}\gamma\, e^{i\omega t} + g_o n\gamma\, e^{2i\omega t}\right] - \frac{\gamma\, e^{i\omega t}}{\tau_\gamma}$$

Drop the second-order nonlinear terms such as g_n, n^2; this procedure also removes terms such as e(2iw). Then cancel the exponentials e(iw) from both sides

$$\begin{cases} i\omega n = -v_g g_o n\bar{\gamma} - v_g \bar{g}\gamma + J - \dfrac{n}{\tau_n} \\ i\omega\gamma = \Gamma v_g g_o n\bar{\gamma} + \Gamma v_g \bar{g}\gamma - \dfrac{\gamma}{\tau_\gamma} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} v_g \bar{g}\gamma = J - \left(v_g g_o \bar{\gamma} + \dfrac{1}{\tau_n} + i\omega \right) n \\ \Gamma v_g g_o \bar{\gamma} n = \left(\dfrac{1}{\tau_\gamma} - \Gamma v_g \bar{g} + i\omega \right) \gamma \end{cases}$$

$$\gamma = \frac{\Gamma v_g g_o \bar{\gamma} J}{\Gamma v_g^2 g_o \bar{g} \bar{\gamma} + \left(v_g g_o \bar{\gamma} + \frac{1}{\tau_n} + i\omega \right) \left(\frac{1}{\tau_\gamma} - \Gamma v_g \bar{g} + i\omega \right)}$$

define

$$\begin{cases} \frac{1}{\tau} = \left(\frac{1}{\tau_\gamma} - \Gamma v_g \bar{g} + v_g g_o \bar{\gamma} + \frac{1}{\tau_n} \right) \\ \omega_o^2 = \Gamma v_g^2 g_o \bar{g} \bar{\gamma} + \left(v_g g_o \bar{\gamma} + \frac{1}{\tau_n} \right) \left(\frac{1}{\tau_\gamma} - \Gamma v_g \bar{g} \right) \end{cases}$$

obtain the transfer function

$$\frac{\gamma}{J} = \frac{\Gamma v_g g_o \bar{\gamma}}{(\omega_o^2 - \omega^2) + (i\omega/\tau)}$$

Defined the response function

$$R_J = \left|\frac{\gamma}{J}\right|^2$$

$$R_J = \frac{\left(\Gamma v_g g_o \bar{\gamma}\right)^2}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\omega^2/\tau^2\right)}$$

peak of the response function

$$\frac{dR_J}{d\omega} = 0$$

$$\omega_r^2 = \omega_o^2 - \frac{1}{2\tau^2}$$

resonant frequency

$$\frac{1}{\tau_\gamma} = v_g(\alpha_{\text{int}} + \alpha_m)$$

نتیجه

Above threshold

$$\frac{1}{\tau_\gamma} = \Gamma v_g \bar{g} \quad \rightarrow \quad \Gamma v_g \bar{g} \bar{\gamma} = \frac{\bar{\gamma}}{\tau_\gamma}$$

$$\omega_{\text{res}} = \sqrt{g_o v_g^2 \bar{\gamma}\left[\alpha_{\text{int}} + \frac{1}{L}\text{Ln}\left(\frac{1}{R}\right)\right]}$$

- Better mirrors lowers B.W.
- Larger differential Gain (a material property) increase BW
- Longer lasers are slower

  Really a trade off according to $\frac{\ln R}{L}$

Classical Electromagnetic and Lasers

By: Dr. Jabbari

Maxwell's Equations and Related Quantities

$$\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t} \qquad \nabla \cdot \vec{\mathscr{D}} = \rho_{\text{free}}$$

$$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t} \qquad \nabla \cdot \vec{\mathscr{B}} = 0$$

J current density

É **electric field**

D **displacement field**

H **magnetic field**

B **Magnetic induction**

P **Charge**

Maxwell's equations

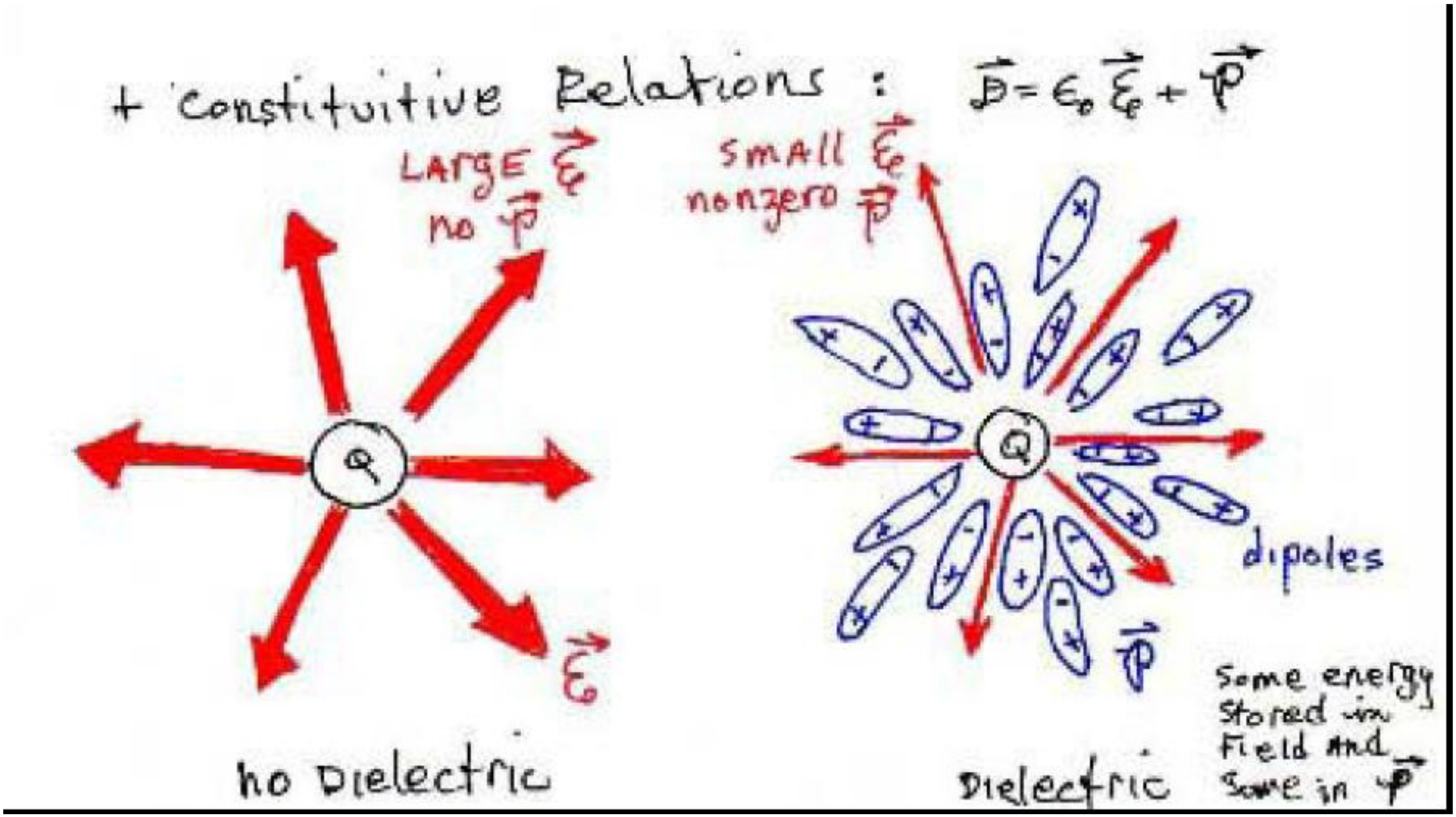
1- find a complex wave vector kn to describe gain/absorption and refractive index

2-find the Poynting vector for electromagnetic power flow

3-develop scattering and linear systems theory for optical devices

4-develop the theory of optical waveguides

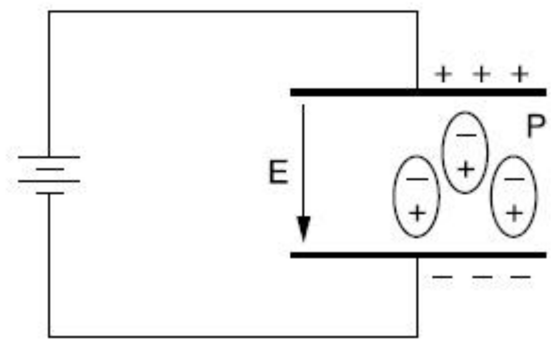
By: Dr. Jabbari


+ Constituitive Relations : $\vec{D} = \epsilon_0 \vec{\mathcal{E}} + \vec{P}$
LARGE $\vec{\mathcal{E}}$
no $\vec{P}$
small $\vec{\mathcal{E}}$
nonzero $\vec{P}$
Q
Q
$\vec{\mathcal{E}}$
dipoles
$\vec{P}$
Some energy
Stored in
Field and
Some in $\vec{P}$
no Dielectric
Dielectric

$$\mathscr{E}(\vec{r},t) = E(\vec{r})e^{i\omega t}$$

ه زمان و مکان است که می توان آن را بطور مستقل از زمان تبدیل کرد

$$\vec{r} = x\tilde{x} + y\tilde{y} + z\tilde{z}$$ → بردار پایه



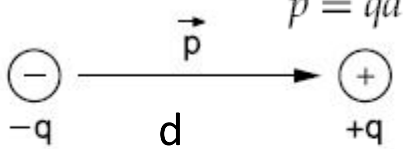
$$\vec{\mathscr{D}} = \varepsilon_0\vec{\mathscr{E}} + \vec{\mathscr{P}}$$

$$\vec{\mathscr{P}}(\vec{r},t)$$

The polarization denotes the **total dipole moment per unit volume** at the position r at time t. The polarization and the dipole moment are related by

$$\vec{P} = \frac{\#\text{dipoles}}{\text{vol}}\,\vec{p}$$

$$\vec{p} = q\vec{d}$$



Linear relation between the induced polarization and the electric field

$$\vec{\mathscr{D}} = \varepsilon_o \vec{\mathscr{E}} + \vec{\mathscr{P}}$$

$$\vec{\mathscr{P}} = \varepsilon_o \chi(\omega) \vec{\mathscr{E}}$$

$$\vec{D} = \epsilon \vec{\xi}$$

permittivity of free space (ε_o)

(complex) susceptibility ($\chi(\omega)$)

$$\varepsilon = \varepsilon_o (1 + \chi)$$

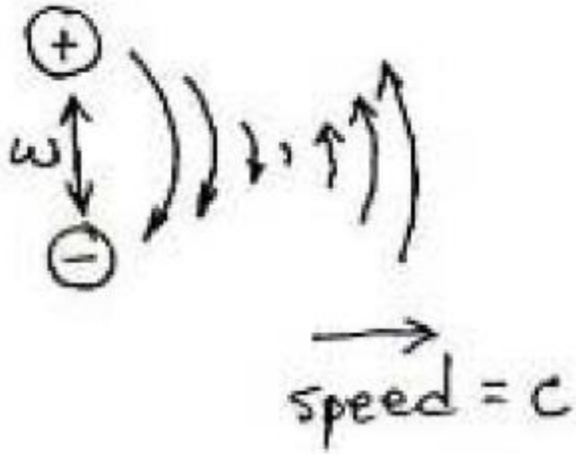
Gauss' law

$$\nabla \cdot \vec{\mathscr{D}} = \rho_{\text{free}} \rightarrow \varepsilon_o \nabla \cdot \vec{\mathscr{E}} + \nabla \cdot \mathscr{P} = \rho_{\text{free}}$$

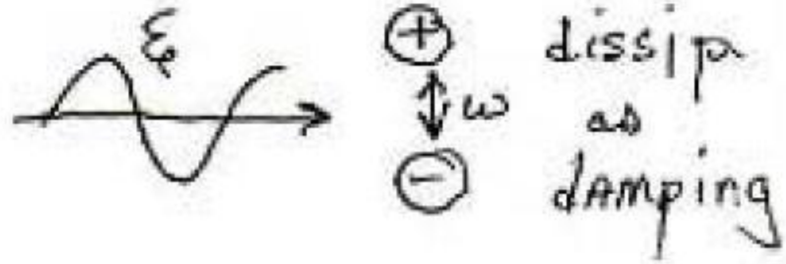
the "**+**" indicates that energy stored as polarization **decreases** the energy stored in the field within a dielectric (since the sum of the two terms equals the **constant free charge**).

- Dipoles emit and absorb E-M waves

Emission



Absorption

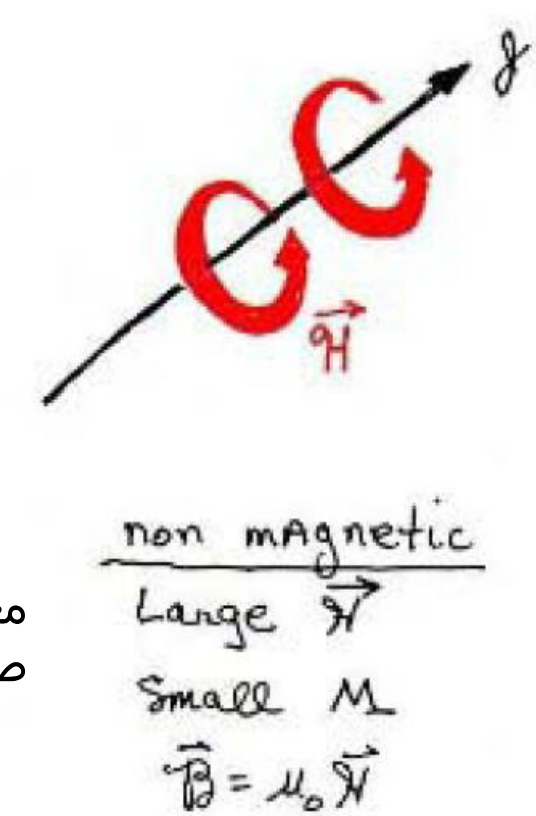


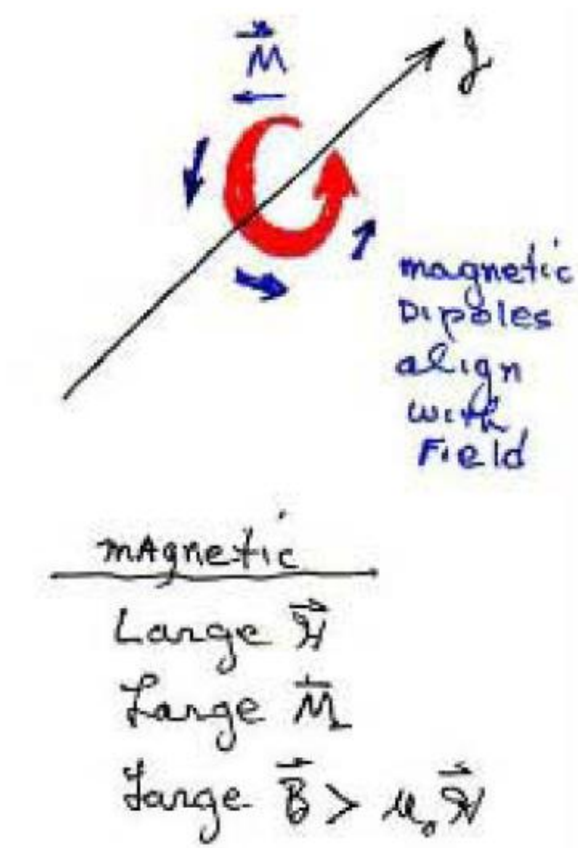
- EM wave interacts with dipoles
  Index of Refraction

relation between the magnetic induction and the magnetic field

$$\vec{\mathscr{B}} = \mu_0 \vec{\mathscr{H}} + \mu_0 \vec{\mathscr{M}}$$

- $\vec{\mathscr{B}}$: magnetic induction
- $\vec{\mathscr{H}}$: magnetic fields
- $\vec{\mathscr{M}}$: material magnetization
- μ_0: permeability of free space





معمولا از M
صرف نظر میگردد

=0 for steady state conditions

$$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t} \rightarrow \nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} \rightarrow \nabla \times \vec{\mathscr{B}} - \mu_o \nabla \times \vec{\mathscr{M}} = \mu_o \vec{\mathscr{J}}$$

current increases, both the magnetization and the magnetic induction also increase

Outside the magnetic material

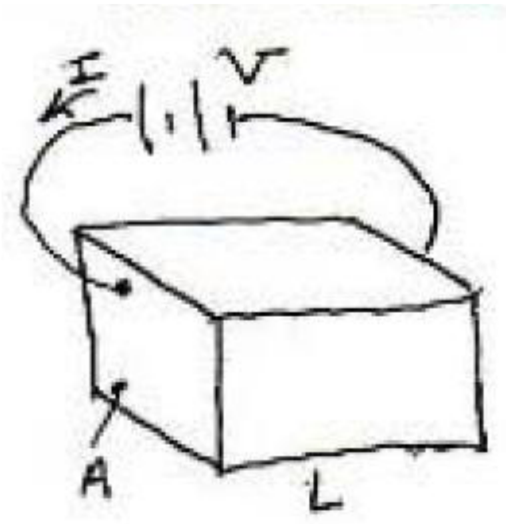
$$\vec{\mathscr{B}} = \mu_o \vec{\mathscr{H}}$$

conductivity of the material

Ohm's law.

$$\vec{\mathscr{J}} = \sigma \vec{\mathscr{E}}$$

Ohm's law.



σ = Conductivity

ρ = Resistivity

$$\sigma = \frac{1}{\rho}$$

$$JA = \sigma \mathcal{E} A = \sigma A \frac{\mathcal{E} L}{L}$$

$$I = \frac{\sigma A}{L} V$$

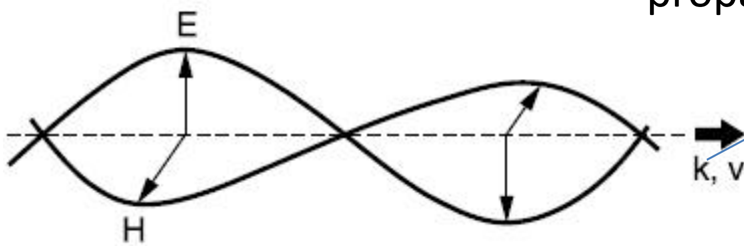
$$R = \frac{L}{\sigma A} = \rho \frac{L}{A}$$

$$J \sim \frac{\text{current}}{\text{area}}$$

Relation between Electric and Magnetic Fields in Vacuum

The vector $\vec{\mathscr{E}} \times \vec{\mathscr{H}}$

propagation direction of the wave



wave vector

We start by plane-wave:

$$\vec{\mathscr{E}}(z,t) = E_o \, e^{ik_o z - i\omega t} \, \tilde{x}$$

$$\vec{\mathscr{H}}(z,t) = H_o \, e^{ik_o z - i\omega t} \, \tilde{y}$$

با استفاده از یکی از معادلات ماکسول شروع می کنیم

$$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t}$$

$$\vec{\mathscr{D}} = \varepsilon_0 \vec{\mathscr{E}} + \vec{\mathscr{P}}$$

$\vec{\mathscr{P}} = 0$ **برای یک فضای آزاد داریم:**

$$\nabla \times \vec{\mathscr{H}} = \varepsilon_0 \frac{\partial \vec{\mathscr{E}}}{\partial t}$$

$$\nabla \times \vec{\mathscr{H}} = \begin{vmatrix} \tilde{x} & \tilde{y} & \tilde{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathscr{H}_x & \mathscr{H}_y & \mathscr{H}_z \end{vmatrix} = \tilde{x}\left(\frac{\partial \mathscr{H}_z}{\partial y} - \frac{\partial \mathscr{H}_y}{\partial z}\right) - \tilde{y}\left(\frac{\partial \mathscr{H}_z}{\partial x} - \frac{\partial \mathscr{H}_x}{\partial z}\right) + \tilde{z}\left(\frac{\partial \mathscr{H}_y}{\partial x} - \frac{\partial \mathscr{H}_x}{\partial y}\right)$$

Only for y-component, we can simplify the relation:

$$\nabla \times \vec{\mathscr{H}} = -\tilde{x}\frac{\partial \mathscr{H}_y}{\partial z} = -i\tilde{x}k_o H_o e^{+ik_o z - i\omega t}$$

The time derivative

$$\frac{\partial}{\partial t}\varepsilon_o \vec{\mathscr{E}}(z,t) = -i\omega\varepsilon_o \tilde{x} E_o\, e^{ik_o z - i\omega t}$$

Substituting

$$k_o H_o = \omega\varepsilon_o E_o \quad \rightarrow \quad H_o = \frac{\omega\varepsilon_o}{k_o}E_o$$

$$B_o = \mu_o H_o$$

$$c^2 = (\varepsilon_o \mu_o)^{-1}$$

$$B_o = \frac{E_o}{c}$$

Relation between Electric and Magnetic Fields in Dielectrics

By the same way:

$$\vec{\mathscr{E}} = E_1\, e^{ikz-i\omega t}\,\tilde{x} \qquad \vec{\mathscr{H}} = H_1\, e^{ikz-i\omega t}\,\tilde{y}$$

$$k = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\lambda_o/n} = k_o n$$

n → real index of refraction

λ_n → in the medium

λ_o → in vacuum

$$\nabla \times \vec{\mathscr{H}} = \frac{\partial}{\partial t}\vec{\mathscr{D}} = \frac{\partial}{\partial t}\left(\varepsilon_o\vec{\mathscr{E}} + \vec{\mathscr{P}}\right) = \frac{\partial}{\partial t}\left(\varepsilon_o\vec{\mathscr{E}} + \varepsilon_o\chi\vec{\mathscr{E}}\right)$$

$$\varepsilon = \varepsilon_o(1+\chi)$$

$$\nabla \times \vec{\mathscr{H}} = \frac{\partial}{\partial t}\left(\varepsilon\vec{\mathscr{E}}\right)$$

The cross product and derivative can be performed in the same manner as above to obtain

$$-i\tilde{x}kH_1\,e^{ikz-i\omega t} = -i\tilde{x}\omega\varepsilon E_1\,e^{ikz-i\omega t}$$

$$H_1 = \frac{\varepsilon\omega}{k_o n}E_1$$

neglects the magnetization

$$M = 0 \quad \Rightarrow \quad B_1 = \frac{\omega\mu_o\varepsilon}{k_o n}E_1$$

$$v = \frac{1}{\sqrt{\mu_o\varepsilon}} = \frac{c}{n} \quad \to \quad \mu_o\varepsilon = \frac{n^2}{c^2}$$

$$c = \frac{\omega}{k_o}$$

$$\Rightarrow \quad B_1 = \frac{c\mu_o\varepsilon}{n}E_1 = \frac{c(n/c)^2}{n}E_1 = \frac{n}{c}E_1 = \frac{E_1}{v}$$

By: Dr. Jabbari

The Wave Equation

Derivation of the Wave Equation

$$\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t}$$

$$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t}$$

$\nabla \times$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = -\frac{\partial}{\partial t}\nabla \times \vec{\mathscr{B}} = -\frac{\partial}{\partial t}\nabla \times (\mu_o \vec{\mathscr{H}})$$

$\vec{\mathscr{J}} = \sigma \vec{\mathscr{E}}$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = -\mu_o \frac{\partial}{\partial t}\left(\vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t}\right) = -\mu_o \frac{\partial \vec{\mathscr{J}}}{\partial t} - \mu_o \frac{\partial^2 \vec{\mathscr{D}}}{\partial t^2} = -\mu_o \sigma \frac{\partial \vec{\mathscr{E}}}{\partial t} - \mu_o \frac{\partial^2}{\partial t^2}\left(\varepsilon_o \vec{\mathscr{E}} + \varepsilon_o \chi \vec{\mathscr{E}}\right)$$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = -\mu_o \sigma \frac{\partial \vec{\mathscr{E}}}{\partial t} - \mu_o \frac{\partial^2}{\partial t^2}\left(\varepsilon \vec{\mathscr{E}}\right)$$

$$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\nabla \times \nabla \times \vec{\mathscr{E}} = \nabla(\nabla \cdot \vec{\mathscr{E}}) - \nabla^2 \vec{\mathscr{E}}$$

$$\nabla^2 \vec{\mathscr{E}} = \mu_o \sigma \frac{\partial \vec{\mathscr{E}}}{\partial t} + \mu_o \varepsilon_o (1 + \chi) \frac{\partial^2 \vec{\mathscr{E}}}{\partial t^2}$$

دانشگاه آزاد اسلامی

The Complex Wave Vector

We start by substituting a plane wave into the wave equation

electric field consists of a single traveling plane wave

$$\vec{\mathscr{E}} = \tilde{e}E_o \exp(ik_n z - i\omega t)$$

$$k_c = k_r + i\,k_i = \mathrm{Re}(k_c) + i\,\mathrm{Im}(k_c)$$

اگر همان مسیری را که برای حالت قبل رفته بودیم در اینجا با بردار موج مختلط انجام دهیم نتیجه زیر بدست خواهد آمد

$$-k_c^2 + i\mu_0\sigma\omega + \mu_0\varepsilon_0\omega^2(1+\chi) = 0$$

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \qquad c = \frac{\omega}{k_o} \rightarrow k_o = \frac{\omega}{c}$$

$$k_c^2 = \frac{\omega^2}{c^2}(1+\chi) + i\mu_o\sigma\omega = k_o^2(1+\chi) + i\mu_o\sigma\omega$$

$$k_c^2 = k_o^2(1+\chi) + i\frac{k_o^2}{k_o^2}\mu_o\sigma\omega = k_o^2(1+\chi) + ik_o^2\frac{\mu_o\sigma\omega}{(\omega/c)^2} = k_o^2(1+\chi) + ik_o^2\frac{\sigma}{\varepsilon_0\omega}$$

$$k_c^2 = k_o^2\left[1+\chi+i\frac{\sigma}{\varepsilon_0\omega}\right]$$

$$k_c = k_o\left[1+\mathrm{Re}(\chi)+i\left(\mathrm{Im}(\chi)+\frac{\sigma}{\varepsilon_0\omega}\right)\right]^{1/2}$$

Definitions for Complex Index, Permittivity and Wave Vector

complex refractive index and the complex permittivity

$$n_c = n_r + i\, n_i$$

$$\varepsilon_c = \varepsilon_r + i\, \varepsilon_i$$

$$k_c = k_o n_c$$

$$n_c^2 = 1 + \mathrm{Re}(\chi) + i\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right)$$

معمولا قسمت حقیقی بدون اندیس نوشته می شود

$$n^2 = \varepsilon/\varepsilon_o \quad \text{or} \quad n = \sqrt{\varepsilon/\varepsilon_o}$$

$$\frac{\varepsilon_r}{\varepsilon_o} + i\frac{\varepsilon_i}{\varepsilon_o} = 1 + \mathrm{Re}(\chi) + i\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right)$$

$$\frac{\varepsilon_r}{\varepsilon_o} = 1 + \mathrm{Re}(\chi) \qquad \frac{\varepsilon_i}{\varepsilon_o} = \mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}$$

So real permittivity and real refractive index are related to the real part of the susceptibility.

imaginary of the permittivity is related to both the imaginary part of the susceptibility and the conductivity

Conductivity → absorbs part of the electromagnetic wave

another definition

$$k_c = k_o n_c = k_o n + i\frac{\alpha}{2} = k_o n - i\frac{g_n}{2}$$

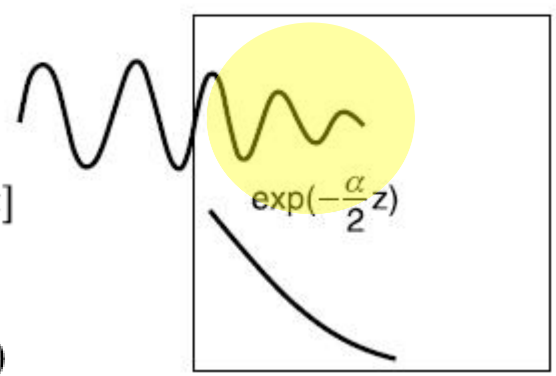
absorption and the gain

The Meaning of k_n

the imaginary part of the wave vector gives the exponential decay or growth (absorption or gain, respectively) of the traveling wave

$$\mathrm{Re}(k_c) = k_o n_r > k_o \quad \rightarrow \quad \lambda_{\text{medium}} < \lambda_{\text{vacuum}}$$

$$E = E_o \exp(ik_c z) = E_o \exp[i(k_o n + i\alpha/2)z)] = E_o \exp(-z\alpha/2)\exp[ik_o nz]$$



The power has the form of $P \sim E^* E \sim \exp(-\alpha z) = \exp(+gz)$

Approximate Expression for the Wave Vector

For simplicity

$$k_c = k_o n_c = k_o\left[1 + \mathrm{Re}(\chi) + i\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right)\right]^{1/2} \Rightarrow k_c = k_o n_c = k_o n_r\left[1 + \frac{i}{n_r^2}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right)\right]^{1/2}$$

$$n_r^2 = 1 + \mathrm{Re}(\chi)$$

$$\sqrt{1+y} \approx 1 - \frac{y}{2}$$

$$k_c = k_o n_c = k_o n_r + i\frac{k_o}{2n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right)$$

$$k_c = k_o n + i\alpha/2$$

very important result

$$\alpha = \frac{k_o}{n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right) = -g_n$$

$$\alpha = -g_n$$

Approximate Expressions for the Refractive Index and Permittivity

$$n_c = \sqrt{\frac{\varepsilon_c}{\varepsilon_o}} = \sqrt{\frac{\varepsilon_r}{\varepsilon_o} + i\frac{\varepsilon_i}{\varepsilon_o}}$$

imaginary part of ε is small

$$\tilde{n} = \sqrt{\frac{\varepsilon_r}{\varepsilon_o}}\left[1 + i\frac{\varepsilon_i/\varepsilon_o}{\varepsilon_r/\varepsilon_o}\right]^{1/2} \cong n_r\left[1 + i\frac{\varepsilon_i/\varepsilon_o}{2\varepsilon_r/\varepsilon_o}\right] = n_r\left[1 + i\frac{\varepsilon_i/\varepsilon_o}{2n_r^2}\right] = n_r + i\frac{\varepsilon_i/\varepsilon_o}{2n_r}$$

$$\tilde{n} = n_r + i\,n_i$$

$$n_r = \sqrt{\frac{\varepsilon_r}{\varepsilon_o}} \qquad n_i = \frac{\varepsilon_i/\varepsilon_o}{2n_r}$$

$$n_r = \sqrt{1 + \mathrm{Re}(\chi)} \qquad n_i = \frac{1}{2n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o\omega}\right)$$

$$\varepsilon_i = \varepsilon_o\ \mathrm{Im}(\chi) + \frac{\sigma}{\omega}$$

The Susceptibility and the Pump

The **real part of** the susceptibility leads to the **index of refraction** while the **imaginary part** leads to **absorption or gain** as can be seen from the main two results

$$n = n_r = [1 + \mathrm{Re}(\chi)]^{1/2}$$

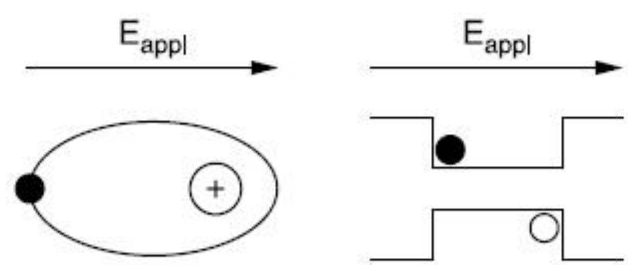
$$\alpha = \frac{k_o}{n_r}\left(\mathrm{Im}(\chi) + \frac{\sigma}{\varepsilon_o \omega}\right) = -g_n$$

The **pump** mechanism adds energy to **the laser**

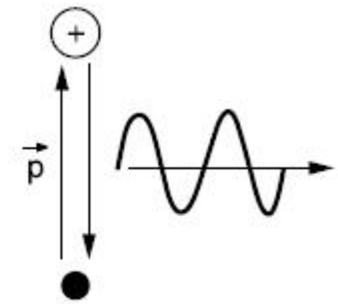
It is the **susceptibility t**hat changes with **pumping** very similar to p**olarization** $P = \varepsilon_o \chi E$

Adding **carriers through** the pump mechanism increases the number of **possible dipoles**

$$\chi = \chi_b + \chi_p.$$



$$\alpha = \frac{k_o}{n_r}\left(\mathrm{Im}(\chi_b) + \mathrm{Im}(\chi_p) + \frac{\sigma}{\varepsilon_o\omega}\right) = \frac{k_o}{n_r}\frac{\sigma}{\varepsilon_o\omega} + \frac{k_o}{n_r}\mathrm{Im}(\chi_b + \chi_p) = \alpha'_{\mathrm{int}} - g$$



related to the term containing the conductivity

related to the term containing the susceptibility

Next

$$n = n_r = \left[1 + \mathrm{Re}(\chi_b) + \mathrm{Re}(\chi_p)\right]^{1/2}$$

Define the background refractive index $n_b = \sqrt{1 + \mathrm{Re}(\chi_b)}.$

مهّم

$$n = n_b\left[1 + \frac{\mathrm{Re}(\chi_p)}{n_b^2}\right]^{1/2} \cong n_b\left[1 + \frac{\mathrm{Re}(\chi_p)}{2n_b^2}\right] = n_b + \frac{\mathrm{Re}(\chi_p)}{2n_b}$$

We see that the refractive **index is smaller than the background** refractive index **when** the real part of the pump susceptibility **is negative**.

the pump susceptibility changes the **index of refraction**

Mathematical Foundations

Linear algebra is the natural mathematical language of **quantum mechanics**

Hilbert spaces for vectors and operators

The Dirac notation

A Hermitian (self-adjoint) operator produces a basis set within a Hilbert space

Observables such as energy or momentum

Vector and Hilbert Spaces

In Q.M.:

Act of observing represented by operators

Particle or system represented by vectors

Exam: ψ (wave funtion) or |ψ> (Ket) represents particle moving along $\hat{z}$ axis unlike Movement *P* . Let *P* be an operator that measures momentum

$$\hat{p}|\psi\rangle = p|\psi\rangle \qquad p = \text{result of measurement}$$

Exam: Alternate statement (Average momentum)

$$\langle\hat{p}\rangle = \text{average momentum} = \langle\psi|\hat{p}|\psi\rangle = p\langle\psi|\psi\rangle = p$$

کمیت (Quantities) مشاهده پذیر (Observable) توسط یک عملگر هرمیتی نمایش داده می شود

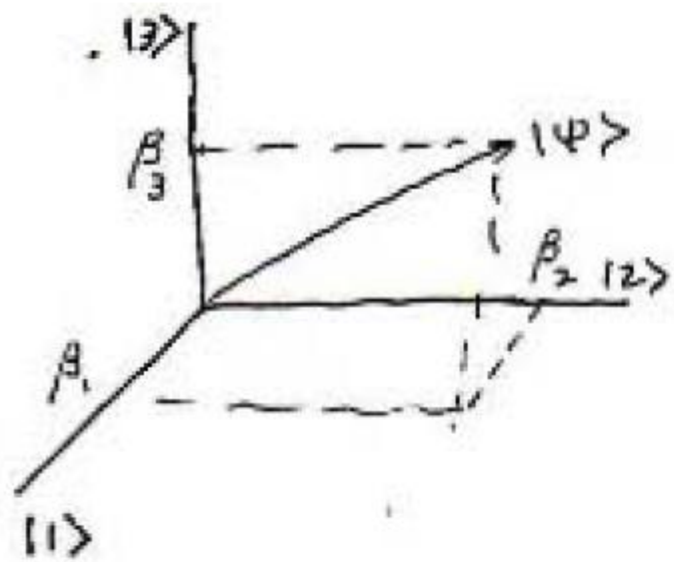
$$\text{Energy} = \hat{H} \qquad \text{momentum} = \hat{p} \qquad \text{Angul. mom } \hat{L}$$

Basic problem in QM. is

Wave functions live in Hilbert space

Hilbert space = vector space + Inner product

There exists basis vectors



$$|\Psi\rangle = \sum_i \beta_i |i\rangle$$

↑ Function (for $|\Psi\rangle$)   ↑ function (for $|i\rangle$)

$$\Psi = \sum_i \beta_i \Psi_i(x)$$

Usually basis set $\{\Psi_i \sim |\Psi_i\rangle \sim |i\rangle\}$ chosen to be eigen vectors of observable like $\mathcal{H}$ ~ Hamilton

$$\hat{O}|\Psi_i\rangle = O_i|\Psi_i\rangle$$

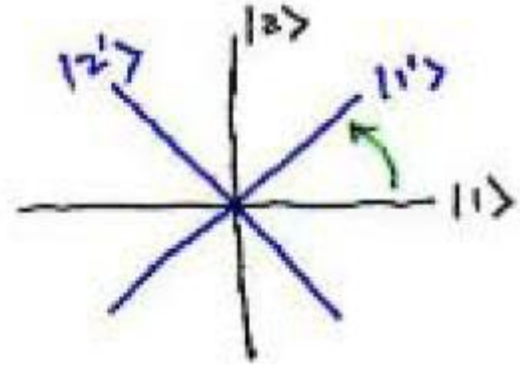
- β_i = component = probability Amplitude

$$= \langle \psi_i | \psi \rangle = \langle i | \psi \rangle$$ = Probability Amplitude

Related to Probability of finding particle with wave function ψ in BASIC State ψ_i

$$P(i) = |\beta_i|^2 = \langle \psi_i | \psi \rangle = \int dx\, \psi_i^* \psi$$

Other type of operator : unitary converts one basis Set into another



+ "Derive" (i.e. Rediscover) Schrodingers Equation

$\hat{\mathcal{H}}\psi = i\hbar\frac{\partial\psi}{\partial t}$: time dependent

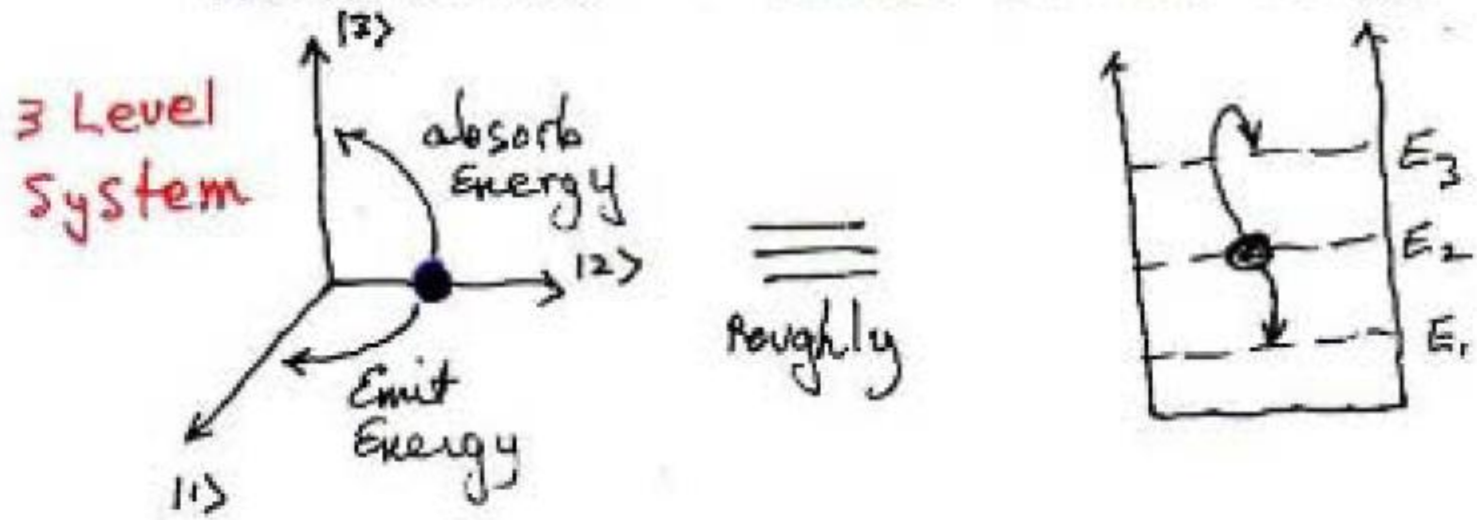
$\hat{\mathcal{H}}\psi_E = E\psi_E$: time independent gives basis vectors

- will need to Review partial Differential Equations

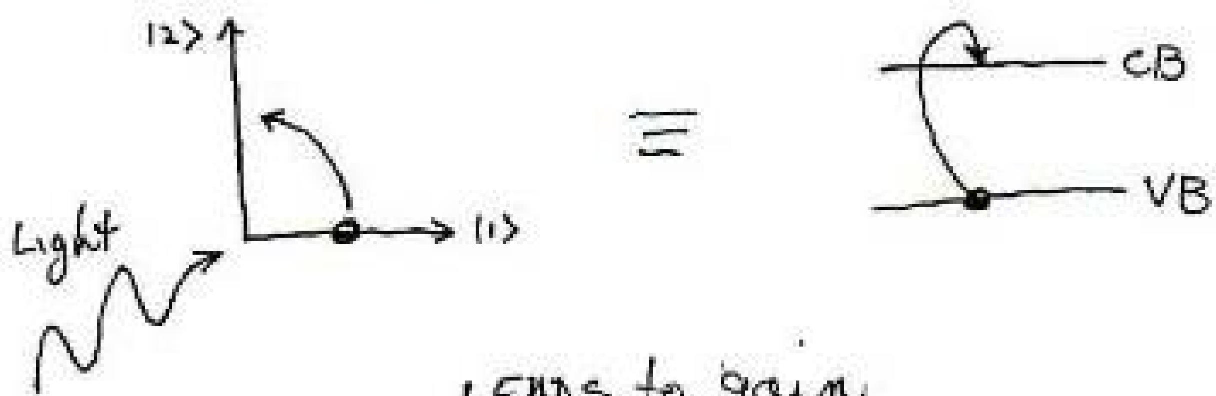
$\hat{\mathcal{H}}\psi = i\hbar\frac{\partial\psi}{\partial t}$

$\hat{\mathcal{H}}\psi_E = E\psi_E$

+ Discuss perturbation theory and transitions : Fermi Golden Rule



+ Two levels



Leads to gain

- + Operators Also form a vector space (and Hilbert Space)
- Operators or Wae functions can carry the dynamics of system
- ·Representations Schrodinger ,Heisenberg ,Interaction

Definition of Vector Space

vector space consists

- set **F**(**real or complex**)with operator $+$ f, f_1, f_2 in F
- Scalar multiplication (SM) over the field of numbers N
  α, β, γ in N

Closure:	$f_1 + f_2$ is in F and αf is in F
Associative:	$(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$
Commutative:	$f_1 + f_2 = f_2 + f_1$
Zero:	There exists a zero vector $\mathcal{O}$ such that $\mathcal{O} + f = f$
Negatives:	For every f in F, there exists $(-f)$ in F such that $f + (-f) = \mathcal{O}$
SM Associative:	$(\alpha\beta)f = \alpha(\beta f)$
SM Distributive:	$\alpha(f_1 + f_2) = \alpha f_1 + \alpha f_2$
SM Distributive:	$(\alpha + \beta)f = \alpha f + \beta f$
SM Unit:	$1f = f$

Example:

If **F** represents the set of **real** functions but the **number** field consists of **complex numbers**

Objects such as c_1 f(x) (where c_1 is complex) cannot be in the original vector space because the function g(x)=c1f(x) has complex values.

closure cannot be satisfied

cannot be vector space

Inner Product, Norm, and Metric

An inner product $\langle \bullet | \bullet \rangle$ in a (real or complex) vector space F is a scalar valued function that maps $F \times F \to C$ (where C is the set of complex numbers) with the properties

1. $\langle f \,|\, g \rangle = \langle g \,|\, f \rangle^*$ with f, g elements in F and where "*" denotes complex conjugate.
2. $\langle \alpha f + \beta g \,|\, h \rangle = \alpha^* \langle f \,|\, h \rangle + \beta^* \langle g \,|\, h \rangle$ and $\langle h \,|\, \alpha f + \beta g \rangle = \alpha \langle h \,|\, f \rangle + \beta \langle h \,|\, g \rangle$ where f, g, h are elements of F and α, β are elements in the complex number field $\mathscr{C}$.
3. $\langle f \,|\, f \rangle \geq 0$ for all vectors f. The inner product can be zero $\langle f \,|\, f \rangle = 0$ if and only if $f = 0$ (except at possibly a few points for functions).

The **norm** or **"length"** of a vector f is defined to be $\|f\| = \langle f \mid f \rangle^{1/2}$

A **metric d(f, g)** is a relation between two elements f and g of a set F such that

1. $d(f, g) \geq 0$ and $d = 0$ only when $f = g$ (except at possibly a few points for piecewise continuous functions $C_p[a,b]$). Recall that two functions are equal only when $f(x) = g(x)$ for all "x" in the domain of definition.
2. $d(f, g) = d(g, f)$.
3. $d(f, g) \leq d(f, h) + d(h, g)$ where h is any third element of F.

The **metric measures** the distance between two elements of the space

$$d(f, g) = \langle f - g \mid f - g \rangle^{1/2}$$

$$\vec{r}_1 = x_1\tilde{x} + y_1\tilde{y}$$

در فضای اقلیدسی R^2

$$\vec{r}_2 = x_2\tilde{x} + y_2\tilde{y}$$

Inner product $\langle \vec{r}_1 \mid . \vec{r}_2 \rangle = \vec{r}_1 \cdot \vec{r}_2 = x_1x_2 + y_1y_2$

Norm $\|\vec{r}_1\| = \langle \vec{r}_1 \mid \vec{r}_1 \rangle^{1/2} = (x_1^2 + y_1^2)^{1/2}$

Metric $d(\vec{r}_1, \vec{r}_2) = \|\vec{r}_1 - \vec{r}_2\| = [(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2)]^{1/2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

inner product for functions

Inner product
Norm

$$\|f(x)\| = \langle f \mid f \rangle^{1/2} = \left[\int_a^b dx\, f(x)^* f(x)\right]^{1/2} = \left[\int_a^b dx\, |f(x)|^2\right]^{1/2}$$

Example: Find the length of f(x)=x for x ε[-1, 1]

$$\|f\| = \langle f \mid f \rangle^{1/2} = \left[\int_{-1}^{1} dx\, x^* x\right]^{1/2} = \left[\int_{-1}^{1} dx\, x \cdot x\right]^{1/2} = \left[\int_{-1}^{1} dx\, x^2\right]^{1/2} = \sqrt{\frac{2}{3}}$$

Kets, Bras, and Brackets for Euclidean Space

3D euclidean vector: $\{\tilde{x}, \tilde{y}, \tilde{z}\}$

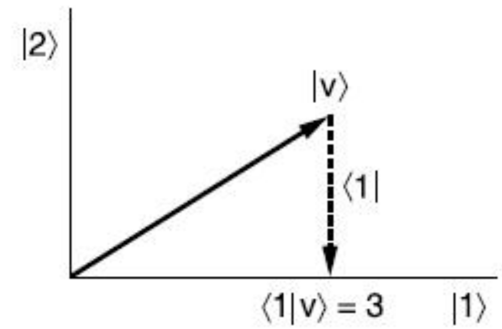
$$\tilde{x} \leftrightarrow |1\rangle \quad \tilde{y} \leftrightarrow |2\rangle \quad \tilde{z} \leftrightarrow |3\rangle \qquad \tilde{e}_n \leftrightarrow |n\rangle$$

Example: $\vec{v} = 3\tilde{x} - 4\tilde{y} + 10\tilde{z}$

$$|v\rangle = 3|1\rangle - 4|2\rangle + 10|3\rangle$$

We define a **"bra" < |** to be a projection operator

$$|v\rangle = 3|1\rangle - 4|2\rangle + 10|3\rangle \Rightarrow \begin{cases} \langle 1|\vec{v} = 3 \\ \langle 2|\vec{v} = -4 \\ \langle 3|\vec{v} = 10. \end{cases}$$



Projection of $\vec{v} = 3\tilde{x} + 5\tilde{y}$ onto $|1\rangle, |2\rangle$.

Combination of projection operators and vectors as

- Already Know $\langle w| \equiv \vec{w}\cdot$

$$\langle 1|\vec{v} = \langle 1|v\rangle$$

"bra" + "ket" gives the "braket" → **inner product**

$$\langle w|v\rangle = (\vec{w}\cdot)\vec{v} = \vec{w}\cdot\vec{v}$$

$|\bullet\rangle\langle\bullet|$ → **to be more complicated compound objects**

Example $\vec{v} = 3\hat{x} + 4\hat{y} - 5\hat{z}$

$$\langle 1|v\rangle = \langle 1|\,|v\rangle = \hat{x}\cdot(3\hat{x}+4\hat{y}-5\hat{z}) = 3$$

or

$$\langle 1|v\rangle = \langle 1|\,|v\rangle = \langle 1|\{3|1\rangle + 4|2\rangle - 5|3\rangle\}$$

$$= 3\langle 1|1\rangle + 4\langle 1|2\rangle - 5\langle 1|3\rangle = 3$$

+ mention Dual Vector Space (more later)

- $\langle w|$ = Linear Operator maps $\mathbb{V} \to \mathbb{C}$

$$\langle w| \sum_{i=1}^{n} c_i |v_i\rangle = \sum_{i=1}^{n} c_i \langle w|v_i\rangle$$

- For every Vector $|w\rangle$ there is an operator $\langle w|$, that is

$$\langle w| \longleftrightarrow |w\rangle \quad \text{or} \quad \vec{W}\cdot \longleftrightarrow \vec{W}$$

Basis, Completeness, and Closure for Euclidean Space

basis set must be orthonormal and complete

$$\langle m|n\rangle = \delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Kronecker delta function

$$\langle 1|2\rangle = 0 = \langle 1|3\rangle = \cdots = \langle m|n\rangle \quad \text{for } m \neq n$$

A linear combination of ''N'' orthonormal vectors $B = \{|1\rangle, |2\rangle, \ldots, |N\rangle\}$ has the form

$$|v\rangle = \sum_{i=1}^{N} C_i |i\rangle$$

C_i can be complex numbers

closure (i.e., completeness) relation

$$\langle m \mid v\rangle = \langle m| \sum_{i=1}^{n} C_i |i\rangle = \sum_{i=1}^{n} C_i \langle m \mid i\rangle = \sum_{i=1}^{n} C_i \delta_{i,m} = C_m$$

$$C_i = \langle i \mid v\rangle$$

$$|v\rangle = \sum_{i=1}^{n} C_i |i\rangle = \sum_{i=1}^{n} [\langle i \mid v\rangle] |i\rangle$$

$$|v\rangle = \sum_{i=1}^{n} |i\rangle\langle i \mid v\rangle \quad \text{or} \quad |v\rangle = \left(\sum_{i=1}^{n} |i\rangle\langle i|\right)|v\rangle$$

$$\sum_{i=1}^{n} |i\rangle\langle i| = 1$$

Example 4.2.1
The completeness relation for $\mathscr{R}^3$ using $\langle w| = \vec{w}\cdot$ is

$$1 = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \quad \text{so} \quad 1 = \tilde{x}\tilde{x}\cdot + \tilde{y}\tilde{y}\cdot + \tilde{z}\tilde{z}\cdot$$

Note that the unit vectors are written next to each other without an operator between them.

The Euclidean Dual Vector Space

Bra projects an arbitrary vector onto the vector

linear operator $\langle w|$ maps a vector space V into the complex numbers $\mathscr{C}$ (i.e., $\langle w|: V \to \mathscr{C}$).

Euclidean vector ,the corresponding bra is the operator $|v\rangle^+ = \langle v| = \vec{v}\cdot$.

$$\langle\cdot| \overset{}{\underset{+}{\leftrightarrow}} |\cdot\rangle \quad \text{or as} \quad |w\rangle^+ = \langle w|$$

"dual vector space V+

$V = \{\, |v\rangle \,\}$ and $V^+ = \{\, \langle w| \,\}$

Example 4.2.2

Find the vector dual to $|2\rangle = \hat{y}$.
The dual vector is $\langle 2| = \hat{y}\cdot$ which is an operator that projects an arbitrary vector $\vec{v}$ onto $\hat{y}$.
We can explicitly represent the result of the projection as the y-component of $\vec{v}$:

$$\langle 2 \mid v\rangle = \hat{y}\cdot\vec{v} = v_y$$

Example 4.2.3

Some relations can be demonstrated for $\vec{v} = |v\rangle = a|1\rangle + b|2\rangle$ where $\{\, |1\rangle, |2\rangle \,\}$ spans R^2.

1. $\langle v| = |v\rangle^+ = [a|1\rangle + b|2\rangle]^+ = \left[|1\rangle^+ a^+ + |2\rangle^+ b^+\right] = a^*\langle 1| + b^*\langle 2|$

2. $\langle v \mid 1\rangle = [a^*\langle 1| + b^*\langle 2|]|1\rangle = a^*$ and $\langle 1 \mid v\rangle = \langle 1|[a|1\rangle + b|2\rangle] = a$

3. $\langle 1 \mid v\rangle = a = (a^*)^* = \langle v \mid 1\rangle^*$ Note that $\langle v \mid 1\rangle^+ = \langle 1 \mid v\rangle = \langle v \mid 1\rangle^*$.

The adjoin reverses the order of operators

$$\langle v|L_1 L_2|w\rangle^+ = \langle w|L_2^+ L_1^+|v\rangle$$

Exam: Assume $\{|i\rangle : \; i = 1, 2, 3\}$

norm

$$\|\vec{v}\|^2 = \langle v \mid v\rangle = \left(\sum_{i=1}^{3} v_i|i\rangle\right)^+ \left(\sum_{j=1}^{3} v_j|j\rangle\right) = \sum_{i=1}^{3} \langle i|v_i^* \sum_{j=1}^{3} v_j|j\rangle = \sum_{i,j=1}^{3} \langle i|v_i^* v_j|j\rangle = \sum_{i,j=1}^{3} v_i^* v_j \langle i \mid j\rangle$$

unit vectors

$$\|v\|^2 = \sum_{i,j=1}^{3} v_i^* v_j \delta_{i,j} = \sum_{i=1}^{3} v_i^* v_i = \sum_{i=1}^{3} |v_i|^2$$

the magnitude of the complex number

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Hilbert Space

A Hilbert space consists of a vector space of functions with a defined inner product

Hilbert Space of Functions with Discrete Basis Vectors

Functions in a set $F = \{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\}$ are **linearly independent** if for complex constants

$$\sum_{i=0}^{n} c_i \phi_i(x) = 0$$

can only be true when all of the complex constants are zero $c_i = 0.$

Functions in the set $F = \{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\}$ are **orthonormal** if $\langle \phi_i \mid \phi_j \rangle = \delta_{ij}$

A linearly independent set of functions F is **complete** if every function f(x) in the space can be written as

$$f(x) = \sum_{i=0}^{n} c_i \phi_i(x) \quad \text{or} \quad |f\rangle = \sum_{i=0}^{n} c_i |\phi_i\rangle$$

the set $\{\phi_i\}$ — complete and orthonormal → can be chosen as basis functions (or **basis vectors**) to span the function space and basis for a **Hilbert space**

$$\{|\phi_0\rangle, |\phi_1\rangle, \ldots\} = \{|0\rangle, |1\rangle, \ldots\}$$

Then we can write each function to form $|f\rangle = \sum_{i=0}^{\infty} c_i |i\rangle$

where the components of the vector can be find as follow:

$$\langle j \mid f\rangle = \langle \phi_j \mid f\rangle = \langle j| \sum_{i=0}^{\infty} c_i |i\rangle = \sum_{i=0}^{\infty} c_i \langle j \mid i\rangle = \sum_{i=0}^{\infty} c_i \delta_{ij} = c_j$$

The **projection** of the function on **the ith axis** produces the inner product between the two complex functions

$$\langle \phi_i \mid f\rangle = \int_a^b dx\, \phi_i^*(x) f(x)$$

$$|f\rangle = \sum_{i=0}^{\infty} c_i |i\rangle \Rightarrow |f\rangle = \sum_{i=0}^{\infty} c_i |i\rangle = \sum_{i=0}^{\infty} \langle i \mid f\rangle |i\rangle = \sum_{i=0}^{\infty} |i\rangle\langle i \mid f\rangle = \left(\sum_{i=0}^{\infty} |i\rangle\langle i|\right)|f\rangle$$

$$\sum_{i=0}^{\infty} |i\rangle\langle i| = 1$$

The **closure relation** ensures **completeness** of the basis set and vice versa

The bra for functions can be written in terms of an operator as

$$\langle f| = \int dx\, f^*(x)\circ$$

The Continuous Basis Set of Functions

$$\langle \phi_K \mid \phi_k\rangle = \delta(k-K)$$

orthonormality

$$\langle f \mid g\rangle = \int dx\, f^*(x) g(x)$$

inner product

$$|f\rangle = \int_a^b dk\, c_k |\phi_k\rangle$$

F has an integral expansion

$$f(x) = \int_a^b dk\, c_k\, \phi_k(x)$$

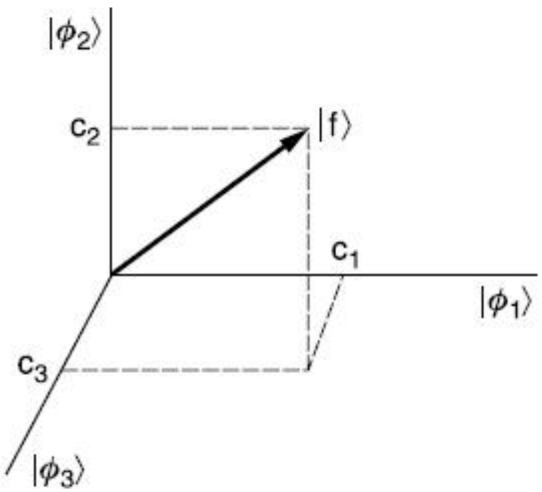

FIGURE 4.3.1
The function f projected onto the basis set of functions.

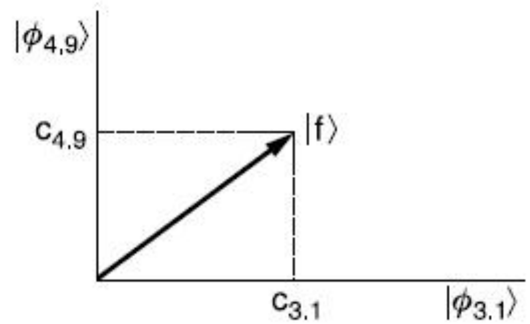

FIGURE 4.3.2
A function projected onto two of the many basis vectors.

Example 4.3.1

Is the set $\{1, x\}$ orthonormal on the interval $[-1, 1]$?

Note that the "1" and "x" represent functions and not coordinates. Therefore, define functions $f = 1$ and $g = x$. These functions are orthogonal on the interval as can be seen

$$\langle f \mid g \rangle = \int_{-1}^{1} dx\, f^* g = \int_{-1}^{1} dx\, 1 \cdot x = 0$$

Neither function is normalized (unit length) since

$$\|f\|^2 = \langle f \mid f \rangle = \int_{-1}^{1} 1\, dx = 2 \quad \text{and} \quad \|g\|^2 = \langle g \mid g \rangle = \int_{-1}^{1} dx\, x^2 = \frac{2}{3}$$

In general, any function $h(x)$ can be normalized by redefining it as $h \to h/\|h\|$. An orthonormal set can be formed by dividing each function by its length. The orthonormal set is

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x \right\}$$

The Sine Basis Set

$$B_s = \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3 \ldots \right\} = \{\psi_n(x) : \ n = 1, 2, 3 \ldots\}$$

$$x \in (0, L)$$

Hilbert space can be expanded every 2L

$$|f\rangle = \sum_{m=1}^{\infty} c_m |\psi_m\rangle \quad \text{or} \quad f(x) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

The expansion coefficients are found by

$$\langle \psi_n \mid f \rangle = \langle \psi_n | \left\{ \sum_m c_m |\psi_m\rangle \right\} = \sum_m c_m \langle \psi_n \mid \psi_m \rangle = c_n$$

$$c_n = \langle \psi_n \mid f \rangle = \left\langle \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \middle| f(x) \right\rangle = \sqrt{\frac{2}{L}} \int_0^L dx\, f(x) \sin\left(\frac{n\pi x}{L}\right)$$

The Cosine Basis Set

$$B_c = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right), \ldots \text{ for } n = 1, 2, 3 \ldots \right\} = \{\phi_0, \phi_1, \ldots\}$$

$$|f\rangle = \sum_{n=0}^{\infty} c_n |\phi_n\rangle$$

$$f(x) = \frac{c_0}{\sqrt{L}} + \sum c_n \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right)$$

$$c_0 = \langle \phi_0 \mid f \rangle = \left\langle \frac{1}{\sqrt{L}} \middle| f(x) \right\rangle = \frac{1}{\sqrt{L}} \int_0^L dx\, f(x)$$

$$c_n = \langle \phi_n \mid f \rangle = \left\langle \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) \middle| f(x) \right\rangle = \sqrt{\frac{2}{L}} \int_0^L dx\, f(x) \cos\left(\frac{n\pi x}{L}\right)$$

The Fourier Series Basis Set

interval (-L, L)

$$B = \left\{ \frac{1}{\sqrt{2L}} \exp\left(i\frac{n\pi x}{L}\right) \quad n = 0, \pm 1, \pm 2 \ldots \right\}$$

orthonormality relation

$$\left\langle \frac{1}{\sqrt{2L}} \exp\left(i\frac{n\pi x}{L}\right) \middle| \frac{1}{\sqrt{2L}} \exp\left(i\frac{m\pi x}{L}\right) \right\rangle = \delta_{nm}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{D_n}{\sqrt{2L}} \exp\left(i\frac{n\pi x}{L}\right)$$

The coefficients D_n can be complex.

The wave is required to repeat itself every length L instead of 2L given above.

$$B = \left\{ \frac{1}{\sqrt{L}} \exp\left(i\frac{2n\pi x}{L}\right) \quad n = 0, \ \pm 1, \ \pm 2 \quad \ldots \right\}$$

For three dimensions

$$B = \left\{ \frac{1}{\sqrt{V}} \exp\left(i\vec{k} \cdot \vec{r}\right) \right\}$$

$$V = L_x L_y L_z \quad \text{and} \quad k_x = (2\pi m/L_x), \ k_y = (2\pi n/L_y), \ k_z = (2\pi p/L_z)$$

The Fourier Transform

$$\left\{\frac{e^{ikx}}{\sqrt{2\pi}}\right\}$$

$$f(x) = \int_{-\infty}^{\infty} dk\, \alpha(k) \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$\left\{ |k\rangle = |\phi_k\rangle = \left| \frac{1}{\sqrt{2\pi}} e^{ik0} \right\rangle \quad \rightarrow \quad \phi_k(x) = \langle x \mid k \rangle = \frac{1}{\sqrt{2\pi}} \exp(ikx) \right\}$$

inner product

$$\langle K \mid k \rangle = \int_{-\infty}^{\infty} dx \frac{e^{-iKx}}{\sqrt{2\pi}} \frac{e^{ikx}}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} dx \frac{e^{i(k-K)x}}{2\pi} = \delta(k - K)$$

The closure relation

$$\hat{1} = \int_{-\infty}^{\infty} |k\rangle dk \langle k|$$

Summary of Results

	Euclidean Vectors	Functions-Discrete Basis	Functions-Continuous Basis
Basis	$\{\|n\rangle :\ n=1,2,3\ldots\} \sim \{\tilde{x},\tilde{y},\tilde{z}\ldots\}$ $n=\text{Integer}$	$\{\|n\rangle = \|u_n\rangle \tilde{u}_n(x)\}$ $n=\text{Integer}$	$\{\|k\rangle = \|\phi_k\rangle \tilde{\phi}_k(x)\}$ $k=\text{Real}$
Projector	$\langle w\| = \vec{w}\cdot$	$\langle f\| = \int dx\, f^*(x) \circ$	$\langle f\| = \int dx\, f^*(x) \circ$
Orthonormality	$\langle m \mid n\rangle = \delta_{m,n}$	$\langle u_m \mid u_n\rangle = \delta_{mn}$	$\langle \phi_K \mid \phi_k\rangle = \delta(k-K)$
Complete	$\|v\rangle = \sum_n c_n \|n\rangle$	$\|f\rangle = \sum_n c_n \|u_n\rangle$ $f(x) = \sum_n c_n u_n(x)$	$\|f\rangle = \int dk\, c_k\, \|\phi_k\rangle$ $f(x) = \int dx\, c_k\, \phi_k(x)$
Components	$c_n = \langle n \mid v\rangle$	$c_n = \langle u_n \mid f\rangle$	$c_k = \langle \phi_k \mid f\rangle$
Inner Product	$\langle v \mid w\rangle = \sum_n v_n^* w_n$	$\langle f \mid g\rangle = \int dx\, f^*(x)\, g(x)$	$\langle f \mid g\rangle = \int dx\, f^*(x)\, g(x)$
Closure	$\sum_n \|n\rangle\langle n\| = \hat{1}$	$\Sigma_n \|u_n\rangle\langle u_n\| = \hat{1}$ $\delta(x-x') = \sum_n u_n^*(x')\, u_n(x)$	$\int dk\, \|\phi_k\rangle\langle\phi_k\| = \hat{1}$ $\delta(x-x') = \int dk\, \phi_k^*(x')\, \phi_k(x)$